



On a total order on the set of Z -numbers based on discrete fuzzy numbers

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Abstract

Z -numbers were introduced by Zadeh in 2011 as a pair of fuzzy numbers (A, B) , where A is interpreted as a fuzzy restriction on the values of a variable, while B is interpreted as a measure of certainty or sureness of A . From the initial proposal, several other approaches have been introduced in order to reduce the computational cost of the involved operations. One of such approaches is called discrete Z -numbers where A and B are modelled as discrete fuzzy numbers. In this paper, the construction of total orders on the set of discrete Z -numbers is investigated for the first time. Specifically, the total order is designed for discrete Z -numbers where the second component has membership values belonging to a finite and prefixed set of values. The method relies on solid and coherent linguistic criteria and several linguistic properties are analyzed. The order involves the transformation of the first components of the discrete Z -numbers by using the credibility of the second components in the sense that a lower credibility enlarges in a greater extent the uncertainty of the first component. Then a total order on the set of discrete fuzzy numbers is applied. Finally, a practical example on how to order discrete Z -numbers is presented and a comparison with other ranking methods is performed from which the strengths of our method are stressed.

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1 Introduction

Computing With Words (CWW) is an interesting hot topic that has not stopped growing (Herrera et al. 2009; Li et al. 2017; Morente-Molinera et al. 2015; Zuheros et al. 2018; Doukas et al. 2010) since the pioneering works by Lotfi Zadeh (1975a,b,c). Among the main reasons for the interest on this research, besides the need to properly model natural language itself, there exist multiple applications that we can find in the literature and that cover many fields such as control systems, decision making problems and consensus process, artificial intelligence, risk assessment, etc. (see Banerjee et al. (2022); Chiclana et al. (2015); Yuan et al. (2023); Nazari-Shirkouhi et al. (2023)). In this way, with the intention of providing linguistic models that describe common natural language sentences such as “Usually, the travel time between Madrid and Barcelona is approximately three hours” or “Very likely, the anticipated budget deficit for the 2018–2019 school year will be reduced closed to 2 million of euros”, Zadeh (2011) introduced the concept of Z-numbers in 2011, as a pair of fuzzy numbers (A, B) , where A is interpreted as a fuzzy constraint on the values that an uncertainty variable X can take, while B is referred as a measure of certainty or sureness of A , referring to the triple (X, A, B) as a Z-valuation. Therefore, if we take into account the two examples we have mentioned above, we have that the associated Z-valuations are $Z_1 = (X = \text{travel time between Madrid and Barcelona, approximately three hours, usually})$ and $Z_2 = (X = \text{the anticipated budget deficit for the 2018–2019 school year, closed to 2 million, very likely})$. Although, as we have already mentioned, the Z-number structure allows modeling many imprecise sentences of natural language, it has the disadvantage of the complexity and the high computational cost of its operations as shown in the literature (Aliiev et al. 2015; Pal et al. 2013; Yager 2012). As it was earlier mentioned by Zadeh, “Problems involving computation with Z-numbers are easy to state but far from easy to solve” (Zadeh 2011). Recently, to overcome the limitations of the traditional approach and with the aim to simplify the computational process, a new perspective on Zadeh’s Z-numbers has been proposed by some authors in Massanet et al. (2016, 2020). In this new approach, Z-numbers are represented by pairs of discrete fuzzy numbers (dfn for short) (A, B) where the second component which determines the degree of sureness (certainty, confidence, reliability, strength of belief) of the first component is not interpreted from a strictly probabilistic point of view, but as an evaluation based on a dfn (Massanet et al. 2014; Riera et al. 2015) facilitating the computations between Z-valuations using aggregation operators on the set of dfns (Casasnovas and Riera 2011; Riera and Torrens 2012, 2014). This new perspective, called discrete Z-numbers, largely resolves the issue of the extremely high computational costs associated with the classical definition of Z-numbers, keeping the established linguistic essence of the original idea proposed by Zadeh.

The research on ordering methods among Z-numbers is a common topic in the literature, mainly due to the multiple applications that this linguistic model has. For instance, in the case of a classical multicriteria decision making problem, a hierarchical order is established among the alternatives according to the value of the collective (or aggregated) linguistic interpretation in order to choose the best alternative (Herrera and Herrera-Viedma 2000).

For this reason, different methods have been proposed by researchers. In this sense, one of the methods is based on transforming the Z -number into a fuzzy number (Bakar and Gegov 2015; Ezadi et al. 2018; Kang et al. 2012). In this way, Bakar and Gegov (2015) use a multi-layer methodology for Z -number classification where the first layer conveys the conversion of the Z -number into a fuzzy number and the second layer provides the classification of the converted Z -number. A similar idea is proposed by Chai et al. (2023) considering an score and accuracy function to each Z -number or by Ezadi et al. (2018) using sigmoid functions. Other proposals are based on the use of Generalized Fuzzy Numbers (GFNs) instead of couples of fuzzy numbers (Chutia 2021; Jiang et al. 2017). Thus, Chutia (2021) ranking method is based on the use of the value and ambiguity index of the GFNs. In this same framework, Jiang et al. (2017) propose an approach based on the centroid points, degrees of fuzziness and spreads of GFNs. The idea of proposing new methods for ranking fuzzy numbers has also led to the construction of methods for ranking Z -numbers. Ezadi and Allahviranloo (2017) propose a new method for ranking fuzzy numbers based on an hyperbolic tangent function and convex combination. Using this same idea, they apply it to rank Z -numbers. Recently, following Yager's ideas to rank interval values based on the golden rule (Yager 2016), Cheng et al. (2022) extend this rule and establish a new Z -number ranking method that preserves the initial information of the Z -number. As discussed in the literature (Cheng et al. 2022), many of the ranking methods based on transforming a Z -number into a fuzzy number lose information in the conversion process and this is mainly due to the fact that a Z -number is interpreted as a single combination of two fuzzy numbers, and the meaning of each of the components that form the Z -number is not taken into account.

This paper proposes for the first time a discrete Z -numbers ranking method based on admissible orders on the set of discrete fuzzy numbers. Admissible orders over the set of closed subintervals of $[0, 1]$ were first investigated by Bustince et al. (2013), and are those total orders that refine the natural order of the set of closed subintervals. Since then, several studies on this topic have been presented and this notion has been adapted to different contexts (De Miguel et al. 2016; Lima et al. 2021; Santana et al. 2020). In the framework of the set of discrete fuzzy numbers, the construction of admissible orders is investigated in Riera et al. (2021). Based on these ideas, this paper aims to construct a total order on the set of discrete Z -numbers. More specifically, the order is established on the subset of discrete Z -numbers where the second component has membership values from a finite and prefixed set of values. Moreover, it is proved that this total order is consistent with the linguistic meaning of the two components of the discrete Z -number according to Zadeh's original proposal, which is one of the main shortcomings that the known orders on Z -numbers in the literature have (Cheng et al. 2022).

The structure of the paper is as follows. In Sect. 2 all the necessary definitions and concepts about orders, discrete fuzzy numbers and Z -numbers to make the paper as self-contained as possible are recalled. In Sect. 3, we compute the total number of discrete fuzzy numbers with a finite set of possible membership values and we introduce the concept of relative rank on these discrete fuzzy numbers. By using these concepts, in Sect. 4 we introduce a new total order defined over the set of discrete Z -numbers and several properties of this order are proved which guarantee that the order has a significant linguistic meaning. Section 5 provides an example to show an application of the order. Finally, in Sect. 6 we perform a comparative study with other ranking methods in the literature.

2 Preliminaries

In order to facilitate a comprehensive understanding of the paper, it is necessary to introduce some preliminary concepts. First of all, let us introduce some basic concepts about orders.

Definition 1 (Bustince et al. (2013)) Given a non-empty set A , a partial order $\preceq$ on the set A is a binary relation on A which is reflexive, antisymmetric and transitive, i.e., a binary relation that satisfies the following properties for each $a, b, c \in A$:

1. $a \preceq a$ (reflexivity).
2. if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry).
3. if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).

We will write $a < b$ if $a \preceq b$ but $a \neq b$. A set A with a partial order $\preceq$ is called a partially ordered set (poset, for short) and denoted by $(A, \preceq)$. If in a poset $(A, \preceq)$ any two elements a, b are comparable, i.e., either $a \preceq b$ or $b \preceq a$ hold, the partial order $\preceq$ is called a linear (or total) order (in this case A is called a chain).

Let us denote by $L([0, 1])$ the set of all closed subintervals of the unit interval, $L([0, 1]) = \{[a, b] \mid 0 \leq a \leq b \leq 1\}$.

Definition 2 (Bustince et al. (2013)) Let $(L([0, 1]), \preceq)$ be a poset. The order $\preceq$ is called an admissible order, if

- (i) $\preceq$ is a total (linear) order on $L([0, 1])$,
- (ii) for all $[a, b], [c, d] \in L([0, 1])$, $[a, b] \preceq [c, d]$ whenever $[a, b] \leq_2 [c, d]$ where $\leq_2$ denotes the classical partial order of intervals $[a, b] \leq_2 [c, d] \iff (a \leq c) \text{ and } (b \leq d)$.

Three classical examples of admissible orders on $L([0, 1])$ are the lexicographic order (lex1), the antilexicographic order (lex2) and the order (XY) proposed by Xu and Yager in (2006).¹

Definition 3 (Bustince et al. (2013); Xu and Yager (2006)) Given $[a, b], [c, d] \in L([0, 1])$, we define:

$$\begin{aligned} [a, b] \leq_{\text{lex1}} [c, d] &\iff a < c \text{ or } (a = c \text{ and } b \leq d), \\ [a, b] \leq_{\text{lex2}} [c, d] &\iff b < d \text{ or } (b = d \text{ and } a \leq c), \\ [a, b] \leq_{\text{XY}} [c, d] &\iff (a + b < c + d) \text{ or } [(a + b = c + d) \text{ and } (b - a \leq d - c)]. \end{aligned}$$

Now let us introduce some concepts related to discrete fuzzy numbers. By a fuzzy set of $\mathbb{R}$, we mean a function $A : \mathbb{R} \rightarrow [0, 1]$. For each fuzzy set A , let $A^\alpha = \{x \in \mathbb{R} \mid A(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$ be its α -cut. The support of a fuzzy set A is the set $\text{supp}(A) = \{x \in \mathbb{R} \mid A(x) > 0\}$. The core of a fuzzy set A is the α -cut A^1 , that is, $\text{core}(A) = \{x \in \mathbb{R} \mid A(x) = 1\}$. In this paper we focus on the discrete version of fuzzy numbers that we call discrete fuzzy numbers.

Definition 4 (Voxman (2001)) A fuzzy set A on $\mathbb{R}$ is called a discrete fuzzy number (dfn) if its support is finite, i.e., there exist $x_1, \dots, x_n \in \mathbb{R}$ with $x_1 < x_2 < \dots < x_n$ such that $\text{supp}(A) = \{x_1, \dots, x_n\}$, and there are natural numbers s, t with $1 \leq s \leq t \leq n$ such that:

¹ The original work of Xu and Yager (2006) defines the order for intuitionistic fuzzy sets but we refer here to its transformation to $L([0, 1])$ which can be found in Bustince et al. (2013).

1. $A(x_i) = 1$ for any natural number i with $s \leq i \leq t$ (core).
2. $A(x_i) \leq A(x_j)$ for each natural number i, j with $1 \leq i \leq j \leq s$.
3. $A(x_i) \geq A(x_j)$ for each natural number i, j with $t \leq i \leq j \leq n$.

A dfn A with $\text{supp}(A) = \{x_1, \dots, x_n\}$ is denoted for short as $A = \{A(x_1)/x_1, \dots, A(x_n)/x_n\}$.

Starting from this point forward, we use the notation L_n to refer to the finite chain $L_n = \{0, 1, \dots, n\}$. The set of all dfns whose support lies within L_n is referred to as $\mathcal{D}_{L_n}$. Additionally, the set of dfns having a closed subinterval of the finite chain L_n as their support is denoted by $\mathcal{A}_1^{L_n}$. The significance of exploring $\mathcal{A}_1^{L_n}$ lies in the ability of this category of discrete fuzzy numbers to serve as linguistic expressions that accurately represent the viewpoints of an expert in a decision-making scenario (Massanet et al. 2014; Riera et al. 2015).

Now, we recall the total order on dfns which will constitute the basis to construct a total order on discrete Z -numbers. Let us start with a preliminary definition:

Definition 5 (Riera et al. (2021)) Let $A \in \mathcal{A}_1^{L_n}$ such that $\text{supp}(A) = [i, j]$, $i \leq j$, $i, j \in L_n$. Then, $\alpha \in (0, 1]$ is called a relevant α -level for A if there exists $x_i \in \text{supp}(A)$ such that $A(x_i) = \alpha$.

Definition 6 (Riera et al. (2021)) Let $A, B \in \mathcal{A}_1^{L_n}$ be two discrete fuzzy numbers whose sets of relevant α -levels are $S_A = \{\alpha_1 < \dots < \alpha_k = 1\}$ with $k \leq n + 1$, $S_B = \{\beta_1 < \dots < \beta_m = 1\}$ with $m \leq n + 1$ respectively, and $S_{AB} = S_A \cup S_B = \{\gamma_1 < \gamma_2 < \dots < \gamma_t = 1\}$ with $1 \leq t \leq k + m - 1$. Let us consider an admissible order $\preceq_\delta$ on $\Pi[L_n] = \{[a, b] : 0 \leq a \leq b \leq n, a, b \in L_n\}$. We will say $A \prec_{\Delta_\delta^\downarrow} B$ if and only if $A \neq B$ and there exists a natural number $j \in I$ such that $A^{\gamma_j} \prec_\delta B^{\gamma_j}$ and $A^{\gamma_i} = B^{\gamma_i}$ for all $i > j$. We will say $A \preceq_{\Delta_\delta^\downarrow} B$ if and only if $A = B$ or $A \prec_{\Delta_\delta^\downarrow} B$.

Remark 1 It is easy to check that $A = B$ if and only if all their level sets in S_{AB} are equal, that is, $A^{\gamma_i} = B^{\gamma_i}$ for all $i \in I = \{1, \dots, t\}$.

Theorem 1 (Riera et al. (2021)) The binary relation $\preceq_{\Delta_\delta^\downarrow}$ is a total order on $\mathcal{A}_1^{L_n}$.

Remark 2 Although the total order recalled in Definition 6 relies on the use of an admissible order on $\Pi[L_n]$, the binary relation $\preceq_{\Delta_\delta^\downarrow}$ remains a total order when considering any total (non necessarily admissible) on $\Pi[L_n]$.

Finally, let us recall the concept of Z -number introduced by Zadeh (2011) as the basis of a new linguistic computational model capable of modeling natural language in a precise way.

Definition 7 (Zadeh (2011)) An ordered pair of fuzzy numbers (A, B) is a Z -number. A Z -number is associated with a real-valued uncertain variable, X , with the first component A , playing the role of fuzzy restriction on X , while the fuzzy number B is an imprecise estimation of reliability of A . The ordered triple, (X, A, B) , is referred as a Z -valuation and it is equivalent to an assignment statement, X is (A, B) .

However, as it is stated in the introduction, the original idea proposed by Zadeh has some drawbacks related to the high computational cost of its associated operations. For this reason, in the literature, several new approaches have been presented which drastically reduce the computational cost. In this way in Massanet et al. (2016) a new vision of Z -numbers where the Z -number is expressed as a pair of discrete fuzzy numbers is proposed.

Definition 8 (Massanet et al. (2016)) Let us consider L_n and L_m two finite chains. An ordered pair of discrete fuzzy numbers (A, B) with $A \in A_1^{L_n}$, $B \in A_1^{L_m}$ is a (L_n, L_m) -discrete Z -number. An (L_n, L_m) -discrete Z -number is associated with an uncertain variable, X , with the first component, A , playing the role of fuzzy restriction on X , while the dfn, B is an imprecise estimation of reliability of A . The ordered triple (X, A, B) is referred as a Z -valuation and it is equivalent to an assignment statement, X is (A, B) .

The set of these discrete Z -numbers is denoted by $\mathcal{Z}(A, B)$ and an arbitrary element will be denoted by $Z(A, B)$.

Unlike Zadeh's Z -numbers, which require integration and calculus to operate, these discrete Z -numbers can be manipulated using simple arithmetic and logical operations (Massanet et al. 2016).

3 On discrete fuzzy numbers on $\mathcal{A}_1^{L_n}$ with membership values from a finite and prefixed set of values

The main goal of the paper is to define a total order on the set of discrete Z -numbers. In fact, we will define a total order on the subset of discrete Z -numbers defined as pairs of discrete fuzzy numbers on $\mathcal{A}_1^{L_n}$ where the second component has membership values from a finite and prefixed set of values. Let us introduce this subset formally.

We will denote by $\mathcal{A}_1^{L_n \times Y_m}$ the subset of discrete fuzzy numbers A with support a closed subinterval of the finite chain L_n and whose membership values $A(x)$ with $x \in L_n$ are in the set $Y_m = \{y_1 = 0, y_2, \dots, y_m = 1\}$, with $0 = y_1 < y_2 < \dots < y_{m-1} < y_m = 1$.

Definition 9 Let $L_{n_A} = \{0, 1, \dots, n_A\}$ and $L_{n_B} = \{0, 1, \dots, n_B\}$ be two finite chains and $Y_m = \{y_1 = 0, y_2, \dots, y_m = 1\}$, with $0 = y_1 < y_2 < \dots < y_{m-1} < y_m = 1$. A second component finite-valued (L_{n_A}, L_{n_B}) -discrete Z -number is a discrete Z -number $Z(A, B)$ with $A \in A_1^{L_{n_A}}$ and $B \in A_1^{L_{n_B} \times Y_m}$.

The set of these discrete Z -numbers is denoted by $\mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ and an arbitrary element will be denoted by $Z(A, B)$.

The main reason to restrict this study to this subset of discrete Z -numbers is that the proposed total order will rely on the existence of a finite number of potential second components of the discrete Z -numbers. This fact, which seems at a first glance a potential shortcoming for applying the order in applications, is not a significant restriction due to two reasons. On the one hand, human experts are not able to differentiate among an infinite number of possible membership values and each expert always considers a finite subset of possible membership values. On the other hand, since the support of the discrete fuzzy numbers is a closed subinterval of the finite chain L_n , each single discrete fuzzy number has a finite set of membership degrees naturally. This implies that in any decision-making problem, the set of membership values considered by the experts for providing their evaluations is finite and this set is going to be the Y_m that will be used for ranking the opinions. Therefore, in real applications, second component finite-valued discrete Z -numbers will not be a restriction at all. In fact, discrete Z -numbers defined as pairs of discrete fuzzy numbers on $\mathcal{A}_1^{L_n}$ where each component has membership values from a finite and prefixed set of values would be even more realistic, but for the sake of generality, we will consider only that the second component is finite-valued since the order that we will define does not need a finite number of potential first components of the discrete Z -numbers.

Our total order which will be introduced in the next section will rely on the computation of the relative rank of a discrete fuzzy number in the finite set $\mathcal{A}_1^{L_n \times Y_m}$. For this reason, in this section we will compute the cardinal of the set $\mathcal{A}_1^{L_n \times Y_m}$ and we will define the relative rank function on this set.

Note that the set $\mathcal{A}_1^{L_n \times Y_m}$ forms a totally ordered lattice with the order defined in Definition 6. Let us compute the cardinality of $\mathcal{A}_1^{L_n \times Y_m}$, that we will denote by $|\mathcal{A}_1^{L_n \times Y_m}|$.

We need first a result on the number of monotone mappings between two finite sets.

Lemma 1 *Let $m_1, m_2 \in \mathbb{N}$. Let $A = \{x_1, \dots, x_{m_1}\}$ with $x_1 < \dots < x_{m_1}$ and $B = \{y_1, \dots, y_{m_2}\}$ with $y_1 < \dots < y_{m_2}$ be finite sets with m_1 and m_2 ordered real numbers, respectively. Then there are exactly $\binom{m_1+m_2-1}{m_2-1}$ increasing (or decreasing) mappings $f: A \rightarrow B$.*

Proof The proof can be found in Lemma 3.2 in Munar et al. (2023) for the case $m_2 \geq m_1$. Let us prove the case $m_1 > m_2$. Since f is monotone, without loss of generality we can assume that it is increasing. Therefore, the mapping f is fully determined by counting the number of times that a value x_i with $1 \leq i \leq m_1$ is assigned to the same value y_j with $1 \leq j \leq m_2$. This combinatorial problem is equivalent to determine in how many ways m_1 equal objects can be distributed into m_2 different sets. The formula to compute this combinatorial problem can be found in Theorem 5.3.2 in Biggs (2002):

$$\binom{m_1 + m_2 - 1}{m_2 - 1}.$$

□

Now, we are in position of providing the cardinal of the set $\mathcal{A}_1^{L_n \times Y_m}$.

Proposition 1 *The number of discrete fuzzy numbers in the set $\mathcal{A}_1^{L_n \times Y_m}$ is*

$$|\mathcal{A}_1^{L_n \times Y_m}| = \binom{n+2m-2}{2m-2} = \frac{(n+2m-2)!}{(2m-2)!n!}.$$

Proof To compute the number of discrete fuzzy numbers in the set $\mathcal{A}_1^{L_n \times Y_m}$, we consider four cases according to the values of the core:

1. The core of the dfn is $\{0, \dots, i\}$, with $i \in \{0, \dots, n-1\}$. Let $\mathcal{A}_{1,1}^{L_n \times Y_m}$ be the subset of these dfns.
2. The core of the dfn is $\{i, \dots, j\}$, with $i \in \{1, \dots, n-1\}$ and $j \in \{i, \dots, n-1\}$. Let $\mathcal{A}_{1,2}^{L_n \times Y_m}$ be the subset of these dfns.
3. The core of the dfn is $\{i, \dots, n\}$, with $i \in \{1, \dots, n\}$. Let $\mathcal{A}_{1,3}^{L_n \times Y_m}$ be the subset of these dfns.
4. The dfn with core $L_n = \{0, \dots, n\}$. Let A_1 be this dfn, that is, $A_1(i) = 1$, for all $i \in L_n$.

It can be easily checked that

- $\mathcal{A}_1^{L_n \times Y_m} = \mathcal{A}_{1,1}^{L_n \times Y_m} \cup \mathcal{A}_{1,2}^{L_n \times Y_m} \cup \mathcal{A}_{1,3}^{L_n \times Y_m} \cup \{A_1\}$.
- $\mathcal{A}_{1,i}^{L_n \times Y_m} \cap \mathcal{A}_{1,j}^{L_n \times Y_m} = \emptyset$, for $i, j = 1, 2, 3$, with $i \neq j$.
- $A_1 \notin \mathcal{A}_{1,i}^{L_n \times Y_m}$, $i = 1, 2, 3$.

Therefore, the cardinal of $\mathcal{A}_1^{L_n \times Y_m}$ is:

$$\left| \mathcal{A}_1^{L_n \times Y_m} \right| = \left| \mathcal{A}_{1,1}^{L_n \times Y_m} \right| + \left| \mathcal{A}_{1,2}^{L_n \times Y_m} \right| + \left| \mathcal{A}_{1,3}^{L_n \times Y_m} \right| + 1.$$

Let us compute $\left| \mathcal{A}_{1,i}^{L_n \times Y_m} \right|$, for $i = 1, 2, 3$.

a) Computation of $\left| \mathcal{A}_{1,1}^{L_n \times Y_m} \right|$:

$$\left| \mathcal{A}_{1,1}^{L_n \times Y_m} \right| = \sum_{i=0}^{n-1} \text{Number of decreasing mappings } f : \{i+1, \dots, n\} \longrightarrow \{y_1, \dots, y_{m-1}\}.$$

Using Lemma 1, we have that the number of decreasing mappings $f : \{i+1, \dots, n\} \longrightarrow \{y_1, \dots, y_{m-1}\}$ is $\binom{n-i+m-2}{m-2}$. So, we have:

$$\left| \mathcal{A}_{1,1}^{L_n \times Y_m} \right| = \sum_{i=0}^{n-1} \binom{n-i+m-2}{m-2} = \binom{m+n-2}{m-2} \sum_{i=0}^{n-1} \frac{\binom{n-i+m-2}{m-2}}{\binom{m+n-2}{m-2}}.$$

Now, using the lookup algorithm (page 36 in Petkovsek et al. (1996)), we have:

$$\left| \mathcal{A}_{1,1}^{L_n \times Y_m} \right| = \binom{m+n-2}{m-2} {}_2F_1 \left[\begin{matrix} -n, 1 \\ 2-m-n \end{matrix}; 1 \right] - 1,$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function defined for $a = -n, b = 1, c = 2-m-n$ and $z = 1$. A full description of hypergeometric series and hypergeometric functions can be found in Bailey (1964). The value of ${}_2F_1 \left[\begin{matrix} -n, 1 \\ 2-m-n \end{matrix}; 1 \right]$ is²:

$${}_2F_1 \left[\begin{matrix} -n, 1 \\ 2-m-n \end{matrix}; 1 \right] = \frac{m+n-1}{m-1}.$$

So,

$$\left| \mathcal{A}_{1,1}^{L_n \times Y_m} \right| = \binom{m+n-2}{m-2} \frac{m+n-1}{m-1} - 1 = \binom{m+n-1}{m-1} - 1.$$

b) Computation of $\left| \mathcal{A}_{1,3}^{L_n \times Y_m} \right|$:

$$\left| \mathcal{A}_{1,3}^{L_n \times Y_m} \right| = \sum_{i=1}^n \text{Number of increasing mappings } f : \{0, \dots, i-1\} \longrightarrow \{y_1, \dots, y_{m-1}\}.$$

Similarly, using again Lemma 1, we have that the number of increasing mappings $f : \{0, \dots, i-1\} \longrightarrow \{y_1, \dots, y_{m-1}\}$ is $\binom{i+m-2}{m-2}$. So, we have:

$$\begin{aligned} \left| \mathcal{A}_{1,3}^{L_n \times Y_m} \right| &= \sum_{i=1}^n \binom{i+m-2}{m-2} = \sum_{i=0}^{n-1} \binom{i+m-1}{m-2} = \sum_{j=0}^{n-1} \binom{n-j+m-2}{m-2} = \left| \mathcal{A}_{1,1}^{L_n \times Y_m} \right| \\ &= \binom{m+n-1}{m-1} - 1. \end{aligned}$$

² See <http://functions.wolfram.com/07.23.03.0003.01>.

c) Computation of $|\mathcal{A}_{1,2}^{L_n \times Y_m}|$:

$$\begin{aligned} |\mathcal{A}_{1,2}^{L_n \times Y_m}| &= \sum_{i=1}^{n-1} \text{Number of increasing mappings } f : \{0, \dots, i-1\} \longrightarrow \{y_1, \dots, y_{m-1}\} \\ &\quad \sum_{j=i}^{n-1} \text{Number of decreasing mappings } f : \{j+1, \dots, n\} \longrightarrow \{y_1, \dots, y_{m-1}\} \\ &= \sum_{i=1}^{n-1} \binom{i+m-2}{m-2} \sum_{j=i}^{n-1} \binom{n-j+m-2}{m-2}. \end{aligned}$$

First of all, using the lookup algorithm we obtain:

$$\begin{aligned} \sum_{j=i}^{n-1} \binom{n-j+m-2}{m-2} &= \sum_{j=0}^{n-1-i} \binom{n-j-i+m-2}{m-2} = \frac{(m+n-i-1)!}{(m-1)!(n-i)!} - 1 \\ &= \binom{m+n-i-1}{m-1} - 1. \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathcal{A}_{1,2}^{L_n \times Y_m}| &= \sum_{i=1}^{n-1} \binom{i+m-2}{m-2} \left(\binom{m+n-i-1}{m-1} - 1 \right) \\ &= \sum_{i=1}^{n-1} \binom{i+m-2}{m-2} \binom{m+n-i-1}{m-1} - \sum_{i=1}^{n-1} \binom{i+m-2}{m-2}. \end{aligned}$$

Using again the lookup algorithm, we obtain:

$$\sum_{i=1}^{n-1} \binom{i+m-2}{m-2} = \binom{m+n-2}{m-1} - 1.$$

The other summand can be computed as follows:

$$\begin{aligned} \sum_{i=1}^{n-1} \binom{i+m-2}{m-2} \binom{m+n-i-1}{m-1} &= \sum_{i=0}^{n-2} \binom{i+m-1}{m-2} \binom{m+n-i-2}{m-1} \\ &= \sum_{i=0}^{n-1} \binom{i+m-1}{m-2} \binom{m+n-i-2}{m-1} \\ &\quad - \binom{m+n-2}{m-2} \\ &= (m-1) \binom{m+n-2}{m-1} \sum_{i=0}^{n-1} \frac{\binom{i+m-1}{m-2} \binom{m+n-i-2}{m-1}}{(m-1) \binom{m+n-2}{m-1}} \\ &\quad - \binom{m+n-2}{m-2} \\ &= (m-1) \binom{m+n-2}{m-1} {}_3F_2 \left[\begin{matrix} m, 1-n, 1 \\ 2, 2-m-n \end{matrix}; 1 \right] \end{aligned}$$

Table 1 Cardinality of $\mathcal{A}_1^{L_n \times Y_m}$ for different values of n and m

m	n	L_n	Y_m	$ \mathcal{A}_1^{L_n \times Y_m} $
3	4	{0, 1, 2, 3, 4}	{ $y_1 = 0, y_2, y_3 = 1$ }	70
4	3	{0, 1, 2, 3}	{ $y_1 = 0, y_2, y_3, y_4 = 1$ }	84
4	5	{0, 1, 2, 3, 4, 5}	{ $y_1 = 0, y_2, y_3, y_4 = 1$ }	462
5	4	{0, 1, 2, 3, 4}	{ $y_1 = 0, y_2, y_3, y_4, y_5 = 1$ }	495
5	6	{0, 1, 2, 3, 4, 5, 6}	{ $y_1 = 0, y_2, y_3, y_4, y_5 = 1$ }	3003
6	5	{0, 1, 2, 3, 4, 5}	{ $y_1 = 0, y_2, y_3, y_4, y_5, y_6 = 1$ }	3003

$$- \binom{m+n-2}{m-2}.$$

The value of the previous hypergeometric function is³:

$${}_3F_2 \left[\begin{matrix} m, 1-n, 1 \\ 2, 2-m-n \end{matrix} ; 1 \right] = \frac{m+n-1}{(m-1)n} \left(\frac{(2m+n-2)!(m-1)!}{((2m-2)!(m+n-1)!} - 1 \right).$$

Collecting the previous results, we get that

$$\begin{aligned} |\mathcal{A}_{1,2}^{L_n \times Y_m}| &= \binom{m+n-2}{m-1} \frac{m+n-1}{n} \left(\frac{(2m+n-2)!(m-1)!}{((2m-2)!(m+n-1)!} - 1 \right) - \binom{m+n-2}{m-2} \\ &\quad - \binom{m+n-2}{m-1} + 1 = \binom{2m+n-2}{2m-2} - 2 \binom{m+n-1}{m-1} + 1. \end{aligned}$$

To sum up, the number of dfns is:

$$\begin{aligned} |\mathcal{A}_1^{L_n \times Y_m}| &= |\mathcal{A}_{1,1}^{L_n \times Y_m}| + |\mathcal{A}_{1,2}^{L_n \times Y_m}| + |\mathcal{A}_{1,3}^{L_n \times Y_m}| + 1 \\ &= \binom{2m+n-2}{2m-2} - 2 \binom{m+n-1}{m-1} + 1 + 2 \binom{m+n-1}{m-1} - 2 + 1 \\ &= \binom{n+2m-2}{2m-2}. \end{aligned}$$

□

In Table 1, we show some examples to see how $|\mathcal{A}_1^{L_n \times Y_m}|$ grows in terms of the values given to m and n .

Once we know the total number of dfns that belong to the set $\mathcal{A}_1^{L_n \times Y_m}$, let us define the so-called relative rank function, which uses the rank of a particular dfn in the set $\mathcal{A}_1^{L_n \times Y_m}$. This function will be essential for the total order proposed in the next section.

Definition 10 Let $\leq$ be a total order on $\mathcal{A}_1^{L_n \times Y_m}$. Let $C_A = \{X \in \mathcal{A}_1^{L_n \times Y_m} \mid X \leq A\}$. We define the function ω that gives the relative rank of a dfn A as follows:

$$\begin{aligned} \omega : \mathcal{A}_1^{L_n \times Y_m} &\longrightarrow (0, 1] \\ A &\longrightarrow \omega(A) = \frac{|C_A|}{|\mathcal{A}_1^{L_n \times Y_m}|} = \frac{|C_A|}{\binom{n+2m-2}{2m-2}}, \end{aligned}$$

where $|C_A|$ denotes the cardinality of C_A .

³ See <http://functions.wolfram.com/07.27.03.0021.01>.

Algorithm 1: Algorithm for generating increasing functions**Input:** y =list of possible values of the membership function, n =Length of the finite chain**Output:** $Func$ =Matrix of generated functions by columns**Function** $increasing.generation(y, n)$:

```

   $Func \leftarrow \text{emptylist};$ 
   $Aux \leftarrow \text{emptylist};$ 
   $n \leftarrow \text{length}(y);$ 
  for  $i \leftarrow 1$  to  $n$  do
     $Aux \leftarrow [Aux, y[1]];$ 
    if  $\text{length}(Aux) = Ldfn$  then
       $Func \leftarrow [Func, Aux];$ 
       $Aux \leftarrow \text{emptylist};$ 
    end
    else
       $Aux2 \leftarrow increasing.generation(y, Ldfn - 1);$ 
       $m \leftarrow \text{row.number}(Aux2);$ 
       $Func \leftarrow [Func, \text{add.column}(y[1], m \text{ times}), Aux2];$ 
    end
     $y \leftarrow y.remove(y[1]);$ 
     $Aux \leftarrow \text{emptylist};$ 
  end
  return  $Func$ ;

```

Note that given a total order on $\mathcal{A}_1^{L_n \times Y_m}$ we know that C_A is not the empty set because $A \in C_A$. Furthermore, C_A is finite because $\mathcal{A}_1^{L_n \times Y_m}$ is a finite set and $C_A \subseteq \mathcal{A}_1^{L_n \times Y_m}$. Therefore, $\omega(A)$ is well defined. Moreover, it satisfies the following property.

Proposition 2 Let $A, B \in \mathcal{A}_1^{L_n \times Y_m}$ and $\preceq$ be a total order on $\mathcal{A}_1^{L_n \times Y_m}$. The function ω defined in Definition 10 satisfies that $A \preceq B$ if, and only if, $\omega(A) \leq \omega(B)$.

Proof Let $A, B \in \mathcal{A}_1^{L_n \times Y_m}$ such that $A \preceq B$. Since $\preceq$ is an order relation, it satisfies the transitivity property, therefore it holds that for all $C \in \mathcal{A}_1^{L_n \times Y_m}$ such that $C \preceq A$ then $C \preceq B$, and this implies $C_A \subseteq C_B$ and then, $\omega(A) \leq \omega(B)$.

Now suppose $\omega(A) \leq \omega(B)$. By definition it can be deduced that $|C_A| \leq |C_B|$, therefore for being $\preceq$ a total order we can deduce that $C_A \subseteq C_B$, this implies $A \in C_B$ and consequently, $A \preceq B$. $\square$

It is clear that if one wants to compute the relative rank of a dfn, all the dfns in $\mathcal{A}_1^{L_n \times Y_m}$ need to be computed. In Algorithm 3 we can see the pseudocode of the construction of every element of $\mathcal{A}_1^{L_n \times Y_m}$. This algorithm uses Algorithms 1 and 2 which generate all the possible increasing and decreasing functions used to generate a dfn.

At this point, we can introduce in Algorithm 4 the algorithm to compute the relative rank of a dfn. Note that this algorithm needs as input the list of all dfns in $\mathcal{A}_1^{L_n \times Y_m}$ increasingly sorted according to a total order $\preceq$ on $\mathcal{A}_1^{L_n \times Y_m}$. This ordered list can be obtained applying any efficient sorting algorithm such as merge sort (based on $\preceq$) to the output of Algorithm 3.

All the necessary functions to generate and order all the dfns in the set $\mathcal{A}_1^{L_n \times Y_m}$ are implemented in the Github repository <https://github.com/aarnaumir/znumbers>. The `readme.md` file contains all the details of such an implementation, as well as examples of how to use these functions.

Algorithm 2: An algorithm for generating decreasing functions**Input:** y =list of possible values of the membership function, n =Length of the finite chain**Output:** $Func$ =Matrix of generated functions by columns

Function decreasing.generation(y, n):

```

   $Func \leftarrow increasing.generation(y, Ldfn);$ 
  if  $Ldfn \neq 1$  then
    for  $i \leftarrow 1$  to  $length(Func)$  do
       $Func[i] \leftarrow reverse(Func[i]);$ 
    end
  end
  return  $Func$ ;

```

Algorithm 3: Algorithm for generating all dfns in $\mathcal{A}_1^{L_n \times Y_m}$ **Input:** y =list of possible values of the membership function, n =Length of the finite chain**Output:** $Gdfn$ =List of generated dfns

Function dfns.generation(y, n):

```

   $Gdfn \leftarrow emptylist;$ 
  if  $n \geq 3$  then
    for  $i \leftarrow 2$  to  $n - 1$  do
       $left.dfn \leftarrow increasing.generation(y, i - 1);$ 
      for  $j \leftarrow i$  to  $n - 1$  do
         $right.dfn \leftarrow decreasing.generation(y, n - j);$ 
        for  $K \leftarrow 1$  to  $length(left.dfn)$  do
          for  $L \leftarrow 1$  to  $length(right.dfn)$  do
             $aux \leftarrow concatenate(left.dfn[K], list\ of\ 1's\ of\ length\ j-i+1, right.dfn[L]);$ 
             $Gdfn \leftarrow [Gdfn, aux];$ 
          end
        end
      end
    end
  end
  for  $i \leftarrow 1$  to  $n - 1$  do
     $right.dfn \leftarrow decreasing.generation(y, n - i);$ 
    for  $j \leftarrow 1$  to  $length(right.dfn)$  do
       $aux \leftarrow concatenate(list\ of\ 1's\ of\ length\ i, right.dfn[j]);$ 
       $Gdfn \leftarrow [Gdfn, aux];$ 
    end
  end
  for  $j \leftarrow 1$  to  $n - 1$  do
     $left.dfn \leftarrow increasing.generation(y, j);$ 
    for  $i \leftarrow 1$  to  $length(left.dfn)$  do
       $aux \leftarrow concatenate(left.dfn[i], list\ of\ 1's\ of\ length\ n-j);$ 
       $Gdfn \leftarrow [Gdfn, aux];$ 
    end
  end
  return  $Gdfn$ ;

```

4 Total order on discrete Z-numbers

Now we have all the necessary background to define our total order on the subset of second component finite-valued discrete Z-numbers. The key idea is to order these discrete Z-numbers by obtaining first discrete fuzzy numbers from them and then applying a total order

Algorithm 4: Algorithm for computing the relative rank ω according to a total order $\preceq$ on $\mathcal{A}_1^{L_n \times Y_m}$

Input: $A = \text{dfn}$ in $\mathcal{A}_1^{L_n \times Y_m}$, $y = \text{list of all dfns in } \mathcal{A}_1^{L_n \times Y_m} \text{ increasingly sorted according to } \preceq$

Output: $\omega(A)$

Function `relative.rank(A,y):`

```

  X ← y[1];
  P ← 1;
  while X ≠ A do
    P ← P + 1;
    X ← y[P];
  end
  return  $\frac{P}{\text{length}(y)}$ ;

```

on discrete fuzzy numbers (for instance the one given in Definition 6). As we will prove later, this order has linguistic sense when an adequate underlying interval order is chosen.

As a first step, let us see how we obtain the discrete fuzzy number that will be used later in the definition of the total order.

Definition 11 Let $Z(A, B) \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$. We define A_B , the transformation of A by B as the function given by:

$$A_B(x) = \begin{cases} A(x) + f(\omega(B))(1 - A(x)), & \text{if } x \in \text{supp}(A), \\ 0 & \text{otherwise,} \end{cases}$$

where $f : (0, 1] \rightarrow [0, 1)$ is a strictly decreasing function such that $f(1) = 0$ and ω is the relative rank in $\mathcal{A}_1^{L_{n_B} \times Y_m}$.

Let us prove that A_B is a discrete fuzzy number satisfying some additional properties.

Theorem 2 Let $Z(A, B) \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ and let A_B be the transformation of A by B . Then $A_B \in \mathcal{A}_1^{L_{n_A}}$ having the same support and core of A . Moreover, it satisfies that for any $\alpha \in (0, 1]$ $A_B^{(1-f(\omega(B)))\alpha + f(\omega(B))} = A^\alpha$.

Proof It is clear that

$$\{x \in L_{n_A} \mid A_B(x) > 0\} = \{x \in \text{supp}(A) \mid A(x) + f(\omega(B))(1 - A(x)) > 0\} = \text{supp}(A), \quad (1)$$

since $f(\omega(B))(1 - A(x)) \geq 0$ for all $x \in L_{n_A}$.

Now let us prove that

$$\{x \in L_{n_A} \mid A_B(x) = 1\} = \text{core}(A). \quad (2)$$

Let $A(x) = 1$, then $A_B(x) = 1 + f(\omega(B))(1 - 1) = 1$. Now suppose that $A_B(x) = 1$ then $1 = A(x) + f(\omega(B))(1 - A(x))$, or, equivalently $A(x)(1 - f(\omega(B))) = 1 - f(\omega(B))$. Since $f(\omega(B)) < 1$, this only holds if $A(x) = 1$.

Now, let us prove that $A_B \in \mathcal{A}_1^{L_{n_A}}$ with $\text{supp}(A_B) = \text{supp}(A)$ and $\text{core}(A_B) = \text{core}(A)$. Consider $x_i, x_j \in L_{n_A}$ such that $\min(\text{supp}(A)) \leq x_i \leq x_j < \min(\text{core}(A))$. Since $A \in \mathcal{A}_1^{L_{n_A}}$, it holds that $A(x_i) \leq A(x_j)$. So,

$$A_B(x_i) = A(x_i)(1 - f(\omega(B))) + f(\omega(B)) \leq A(x_j)(1 - f(\omega(B))) + f(\omega(B)) = A_B(x_j).$$

On the other hand, consider $x_i, x_j \in L_{n_A}$ such that $\max(\text{core}(A)) < x_i \leq x_j \leq \max(\text{supp}(A))$. Since $A \in \mathcal{A}_1^{L_{n_A}}$, it holds that $A(x_i) \geq A(x_j)$. So,

$$A_B(x_i) = A(x_i)(1 - f(\omega(B))) + f(\omega(B)) \geq A(x_j)(1 - f(\omega(B))) + f(\omega(B)) = A_B(x_j).$$

Taking into account Eqs. (1) and (2), the result holds.

Finally, let us see that for any $\alpha \in (0, 1]$, it holds $A_B^{(1-f(\omega(B)))\alpha+f(\omega(B))} = A^\alpha$. Since $1 - f(\omega(B)) > 0$, we have that:

$$\begin{aligned} A_B(x) &\geq (1 - f(\omega(B)))\alpha + f(\omega(B)) && \Longleftrightarrow \\ A(x)(1 - f(\omega(B))) + f(\omega(B)) &\geq (1 - f(\omega(B)))\alpha + f(\omega(B)) && \Longleftrightarrow \\ A(x)(1 - f(\omega(B))) &\geq (1 - f(\omega(B)))\alpha && \Longleftrightarrow \\ A(x) &\geq \alpha \end{aligned}$$

and therefore $A_B^{(1-f(\omega(B)))\alpha+f(\omega(B))} = A^\alpha$. $\square$

The transformation of A by B does not only maintain the same core and support of the initial discrete fuzzy number A . As it can be seen in its definition, the positive value $f(\omega(B))(1 - A(x))$ is added to the membership of every element of the support of A . This value is higher as smaller is the relative rank of B in $L_{n_B} \times Y_m$, i.e., as smaller is the reliability/confidence of A . In this way, this transformation changes the uncertainty of A (increasing the membership values outside of the core) according to the reliability/confidence of A provided by B .

Let us illustrate this idea with some examples.

Example 1 In both cases, let $Z(A, B) \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ be a discrete second component finite-valued Z -number with $n_A = n_B = 5$, $m = 4$ and $Y_m = \{0, 0.33, 0.66, 1\}$. The number of discrete fuzzy numbers in $\mathcal{A}_1^{L_{n_B} \times Y_m}$ is $\binom{5+2 \cdot 4 - 2}{2 \cdot 4 - 2} = \binom{11}{6} = 462$ and we will use function $f(x) = 1 - x$ which satisfies the conditions of Definition 11.

- a) Consider $A = \{0.2/1, 0.6/2, 1/3, 1/4, 0.6/5\}$ and $B = \{0.33/3, 1/4, 1/5\}$. By considering the total order $\preceq_{\Delta_\delta^1}$ given in Definition 6 generated from the interval order lex_1 , B is in position 440 and $\omega(B) = \frac{440}{462} \approx 0.9523$. In this case, A_B is given by:

$$A_B = \{0.2380952/1, 0.6190476/2, 1/3, 1/4, 0.6190476/5\},$$

which is quite similar to A since the confidence/reliability of A given by B was high.

- b) Consider now $A = \{0.6/1, 0.8/2, 1/3, 1/4, 0.3/5\}$ and $B = \{1/5\}$. By considering the same total order used in the previous case, B is in position 462 and therefore $\omega(B) = \frac{462}{462} = 1$. The value of A_B is:

$$A_B = \{0.6/1, 0.8/2, 1/3, 1/4, 0.3/5\},$$

which is equal to A since the confidence is the highest possible.

Now, that we have already introduced the background necessary to define the total order on the set $\mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$, we define the binary relation as follows:

Definition 12 Consider two total orders $\preceq_A$ on $\mathcal{A}_1^{L_{n_A}}$ and $\preceq_B$ on $\mathcal{A}_1^{L_{n_B} \times Y_m}$. We define the binary relation on $\mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ associated to $\preceq_A$ and $\preceq_B$ as follows: given $Z(A, B), Z(A', B') \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$,

$$Z(A, B) \preceq Z(A', B') \Longleftrightarrow (A_B \prec_A A'_{B'}) \text{ or } (A_B = A'_{B'} \text{ and } B \preceq_B B').$$

This binary relation works as follows. Given two second component finite-valued discrete Z -numbers $Z(A, B)$ and $Z(A', B')$, we first compare A_B and $A'_{B'}$ because these two discrete fuzzy numbers represent, in each case, the information given by the fuzzy restriction over the variable that we want to model transformed by means of the reliability of this fuzzy restriction. In the case where these two dfns are equal, we consider more relevant the one with a higher reliability.

To prove that this binary relation on $\mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ is a total order, we need the following lemma.

Lemma 2 *Let $Z(A, B), Z(A', B') \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ be such that $A_B = A'_{B'}$ and $B = B'$. Then, it holds that $Z(A, B) = Z(A', B')$.*

Proof First, by Theorem 2, it can be deduced that $\text{supp}(A) = \text{supp}(A')$ because

$$\text{supp}(A) = \text{supp}(A_B) = \text{supp}(A'_{B'}) = \text{supp}(A').$$

Therefore, let us consider $x \in \text{supp}(A)$ and let us see that $A(x) = A'(x)$ to prove the lemma.

Since $B = B'$, we have $\omega(B) = \omega(B')$. Also, since $A_B(x) = A'_{B'}(x)$ for all $x \in \text{supp}(A)$ we have:

$$A(x) + f(\omega(B))(1 - A(x)) = A_B(x) = A'_{B'}(x) = A'(x) + f(\omega(B))(1 - A'(x)).$$

Therefore,

$$(1 - f(\omega(B)))A(x) = (1 - f(\omega(B)))A'(x).$$

Taking into account that $f(\omega(B)) < 1$, this equation holds if $A(x) = A'(x)$. Consequently, $A = A'$ and $Z(A, B) = Z(A', B')$. $\square$

Theorem 3 *The binary relation given in Definition 12 is a total order on the set $\mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$.*

Proof First of all, let us see that the relation defined in Definition 12 is an order relation, that is, it satisfies the reflexivity, antisymmetry and transitivity properties:

- Reflexivity:

Let $Z(A, B) \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$. Since $A_B = A_B$ and $B \preceq_B B$ (in fact, $B = B$), by definition $Z(A, B) \preceq Z(A, B)$.

- Antisymmetry:

Suppose $Z(A, B) \preceq Z(A', B')$ and $Z(A', B') \preceq Z(A, B)$ and consider the following cases:

- $A_B = A'_{B'}$:

Since $Z(A, B) \preceq Z(A', B')$ we have that $B \preceq_B B'$ and since $Z(A', B') \preceq Z(A, B)$ we have that $B' \preceq_B B$. Using the antisymmetry of the order relation $\preceq_B$, we have that $B = B'$ and by Lemma 2, $Z(A, B) = Z(A', B')$.

- $A_B \neq A'_{B'}$:

Since $Z(A, B) \preceq Z(A', B')$ we have that $A_B \prec_A A'_{B'}$. Since $Z(A', B') \preceq Z(A, B)$ we have $A'_{B'} \prec_A A_B$. But this is a contradiction, so it cannot hold that $A_B \neq A'_{B'}$.

- Transitivity:

Suppose $Z(A, B) \preceq Z(A', B')$ and $Z(A', B') \preceq Z(A'', B'')$ and consider the following cases:

- $A_B = A'_{B'} = A''_{B''}$: It holds that $B \preceq_B B' \preceq_B B''$, which implies $A_B = A''_{B''}$ and $B \preceq_B B''$ using the transitivity of order $\preceq_B$.
- $A_B = A'_{B'} \neq A''_{B''}$: In this case, $A'_{B'} \prec_A A''_{B''}$ which implies $A_B = A'_{B'} \prec_A A''_{B''}$.
- $A_B \neq A'_{B'} = A''_{B''}$: This implies $A_B \prec_A A'_{B'} = A''_{B''}$, and consequently, $A_B \prec_A A''_{B''}$.
- $A_B \neq A'_{B'} \neq A''_{B''}$: Now we have that $A_B \prec_A A'_{B'} \prec_A A''_{B''}$, and consequently $A_B \prec_A A''_{B''}$ using the transitivity of order $\preceq_A$.

In any case, $Z(A, B) \preceq Z(A'', B'')$.

It remains to be proved that it is a total order, that is, given $Z(A, B), Z(A', B') \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$, it holds that $Z(A, B) \preceq Z(A', B')$ or $Z(A', B') \preceq Z(A, B)$. So, suppose $Z(A', B') \not\preceq Z(A, B)$, and we will see that $Z(A, B) \preceq Z(A', B')$. Consider two cases:

- $A_B \neq A'_{B'}$:
In this case, since $Z(A', B') \not\preceq Z(A, B)$ we have $A'_{B'} \not\prec_A A_B$. Using that $\preceq_A$ is a total order, it can be deduced that $A_B \prec_A A'_{B'}$ and therefore $Z(A, B) \preceq Z(A', B')$.
- $A_B = A'_{B'}$:
In this case, we consider two more subcases:
 - $B = B'$: Using Lemma 2, we have $Z(A, B) = Z(A', B')$, a contradiction with the hypothesis $Z(A', B') \not\preceq Z(A, B)$.
 - $B \neq B'$: Since $Z(A', B') \not\preceq Z(A, B)$ and $B \neq B'$, we have $B' \not\prec_B B$. Using that $\preceq_B$ is a total order, it can be deduced that $B \prec_B B'$ and therefore $Z(A, B) \preceq Z(A', B')$.

□

Let us see an example illustrating the operating of the order. In some of the cases, the transformations of the first component by the second component coincide and we only take on account the second component of the discrete Z -numbers.

Example 2 Let us consider the set of discrete second component finite-valued Z -numbers $\mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ with $n_A = n_B = 5$, $m = 4$ and $Y_m = \{0, 0.33, 0.66, 1\}$. Consider now the following discrete fuzzy numbers

$$\begin{aligned}
 A &= \{0.2/1, 0.6/2, 1/3, 1/4, 0.6/5\}, \\
 A' &= \{0.6/1, 0.8/2, 1/3, 1/4, 0.3/5\}, \\
 A'' &= \{1/3, 1/4\}, \\
 A''' &= \{0.2380952/1, 0.6190476/2, 1/3, 1/4, 0.6190476/5\}, \\
 B &= \{0.33/3, 1/4, 1/5\}, \\
 B' &= \{1/5\}.
 \end{aligned}$$

The number of discrete fuzzy numbers in $\mathcal{A}_1^{L_{n_B} \times Y_m}$ is $\binom{5+2 \cdot 4 - 2}{2 \cdot 4 - 2} = \binom{11}{6} = 462$. By considering the total order $\preceq_{\Delta_\delta^\downarrow}$ given in Definition 6 generated from the interval order lex_1 as $\preceq_B$, we have that B is in position 440, $\omega(B) = \frac{440}{462} \approx 0.952381$ and B' is in position 462, $\omega(B') = \frac{462}{462} = 1$.

Consider now the function $f(x) = 1 - x$ which satisfies the conditions of the Definition 11 and the total order $\preceq_{\Delta_\delta^\downarrow}$ given in Definition 6 generated from the interval order lex_1 as $\preceq_A$.

a) It holds that $Z(A', B') \prec Z(A, B)$ because A_B is given by:

$$A_B = \{0.2380952/1, 0.6190476/2, 1/3, 1/4, 0.6190476/5\},$$

and $A'_{B'}$ is:

$$A'_{B'} = \{0.6/1, 0.8/2, 1/3, 1/4, 0.3/5\},$$

and $A'_{B'} \prec_{\Delta_\delta^\downarrow} A_B$.

- b) It holds that $Z(A'', B) \prec Z(A'', B')$. Let us check it. In this case, we have that $A''_B = A'' = A'' = A''_{B'}$ since A'' only has elements on the core. Therefore we must consider the total order on the second components to order these discrete Z -numbers. Indeed, it is immediate to see that $B \prec_{\Delta_\delta^\downarrow} B'$.
- c) It holds that $Z(A, B) \prec Z(A''', B')$. Again we have that

$$A_B = A''' = A'''_{B'}$$

but this case is different from the previous one because one of the transformations A_B is different from A . Anyway, we must resort to the total order on the second components to order these discrete Z -numbers. Again, we have that $B \prec_{\Delta_\delta^\downarrow} B'$.

Although we have already proved that the order defined in Definition 12 is a total order, it is of great interest to investigate if this order has a linguistic meaning, that is, if it presents a behavior which is compatible with the intuition of a user when sorting a set of discrete Z -numbers.

Let us start with a first lemma which will be useful for the subsequent results.

Lemma 3 *Let $Z(A, B), Z(A', B) \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ be two discrete Z -numbers with the same second component. If $\alpha = \max\{\gamma \in (0, 1] \mid A^\gamma \neq A'^\gamma\}$, then $\alpha_B := \alpha + f(\omega(B))(1 - \alpha) = \max\{\beta \in (0, 1] \mid A^\beta_B \neq A'^\beta_B\}$.*

Proof Since according to Theorem 2, for all $\alpha \in (0, 1]$, $A_B^{(1-f(\omega(B)))\alpha+f(\omega(B))} = A^\alpha$, the proof is straightforward. $\square$

Let us introduce two linguistic properties that, in our opinion, it is advisable that any total order on discrete Z -numbers fulfill in order to have linguistic meaning. First of all, given two discrete Z -numbers $Z(A, B), Z(A', B') \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$, a desirable linguistic property would be that when the values of the first component are the same, that is, $A = A'$, i.e., the fuzzy restrictions over the real-valued uncertain variable X coincide, these discrete Z -numbers should be ranked according to the degree of certainty/reliability of such restriction, in the sense that the one with the higher certainty is greater. Formally, this is translated to:

$$\text{If } A = A', \text{ then } B \leq_B B' \iff Z(A, B) \leq Z(A', B').$$

On the other hand, if the values of the second component are the same, that is, $B = B'$, we have that the degrees of certainty/reliability are the same, so we only have to take into account the order on the values of the first component to compare the two discrete Z -numbers. Formally, this is translated to:

$$\text{If } B = B', \text{ then } A \leq_A A' \iff Z(A, B) \leq Z(A', B').$$

We have to take into account that we have introduced a whole family of total orders on the set $\mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ since this total order is based on two total orders on the sets $\mathcal{A}_1^{L_{n_A}}$ and $\mathcal{A}_1^{L_{n_B}}$, which in turn, depend on the choice of admissible orders within the set of intervals of L_{n_A} and L_{n_B} . Unfortunately, the two properties do not hold for any configuration, but we will see that certain configurations fulfill both properties.

Given two discrete Z -numbers $Z(A, B), Z(A', B') \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$, notice that, by construction, if $A = A'$ and $B \leq_B B'$, we have that for all $\alpha \in (0, 1]$, $A_{B'}^\alpha \subseteq A_B^\alpha$. This fact tells us that if we want $B \leq_B B' \iff Z(A, B) \leq Z(A, B')$, we need to choose a total order within the set of intervals of L_{n_A} where being contained is equivalent to being greater. This means that we require an order that satisfies that if $A_{B'}^\alpha = [x_1, x_2]$ and $A_B^\alpha = [x_3, x_4]$ such that $x_1 \leq x_3$ and $x_4 \leq x_2$, then the interval $[x_3, x_4]$ must be greater. This can be achieved considering a total order that extends the following partial order defined over the intervals.

Definition 13 Given $[a, b], [c, d] \in \Pi[L_n]$ we define the binary relation $\leq_{\text{inc}}$ on the set $\Pi[L_n]$ as:

$$[a, b] \leq_{\text{inc}} [c, d] \iff (a \leq c \text{ and } d \leq b)$$

It is clear that the binary relation defined in Definition 13 is a partial order on the set $\Pi[L_n]$. This order ensures that the interval-inclusion property is satisfied, in the sense that if an interval is contained within another interval, it is considered to be greater with respect to the interval ordering. Now we will search for a total order that fulfills the interval-inclusion property, that is an order that refines the partial order defined in Definition 13. We understand that a total order refines a given partial order as whenever two elements are comparable in the partial order, they are compared in the same way by using the total order. Looking at the definition of an admissible order defined over the intervals in Bustince et al. (2013) one can make a reconstruction of the definition based on the one defined in Definition 13 in the following way.

Definition 14 Let $\leq_{\text{inc}}$ be the partial order defined in Definition 13. We say that the binary relation $\leq$ on the set $\Pi[L_n]$ is an admissible order with respect to $\leq_{\text{inc}}$ if it fulfills the following properties:

1. It is a total order.
2. It refines $\leq_{\text{inc}}$, i.e., for any $[a, b], [c, d] \in \Pi[L_n]$ such that $[a, b] \leq_{\text{inc}} [c, d]$, it holds that $[a, b] \leq [c, d]$.

Let us define an interval order that fulfills the properties of Definition 14 to see an example of an admissible order with respect to $\leq_{\text{inc}}$.

Definition 15 Given $[a, b], [c, d] \in \Pi[L_n]$ we define the binary relation $\leq_{\text{t-inc}}$ on the set $\Pi[L_n]$ as:

$$[a, b] \leq_{\text{t-inc}} [c, d] \iff a < c \text{ or } (a = c \text{ and } d \leq b).$$

It is a matter of straightforward computation to check that $\leq_{\text{t-inc}}$ is a total order on $\Pi[L_n]$ and that refines $\leq_{\text{inc}}$.

Now, let us prove the two linguistic properties.

Theorem 4 Let $Z(A, B), Z(A', B') \in \mathcal{Z}(L_{n_A}, L_{n_B} \times Y_m)$ and consider $\leq$ the total order defined in Definition 12 generated from the total orders $\leq_{\Delta_A^\downarrow}$ on $\mathcal{A}_1^{L_{n_A}}$ (with underlying total order $\leq_{\delta_A}$ on $\Pi[L_{n_A}]$) and $\leq_{\Delta_B^\downarrow}$ on $\mathcal{A}_1^{L_{n_B}}$ (with underlying total order $\leq_{\delta_B}$ on $\Pi[L_{n_B}]$). The following statements are true.

- a) If $A = A'$ and $\leq_{\delta_A}$ is an admissible order with respect to $\leq_{\text{inc}}$ in the sense of Definition 14, then $B \leq_{\Delta_B^\downarrow} B' \iff Z(A, B) \leq Z(A', B')$.

b) If $B = B'$ then $A \preceq_{\Delta_{\delta_A}^\downarrow} A' \iff Z(A, B) \preceq Z(A', B)$.

Proof a) Suppose $A = A'$, and consider two cases:

- $A_B = A'_{B'}$: In this case, by the definition of $\preceq$, it holds that $B \preceq_{\Delta_{\delta_B}^\downarrow} B' \iff Z(A, B) \preceq Z(A', B')$.
- $A_B \neq A'_{B'}$: Since $A_B \neq A'_{B'}$, one easily sees that $B \neq B'$. Call $\lambda = f(\omega(B)) - f(\omega(B')) \neq 0$. For any $x \in \text{supp}(A) \setminus \text{core}(A)$ such that $A_B(x) \neq A'_{B'}(x)$, it holds that

$$A_B(x) - A'_{B'}(x) = (1 - A(x))(f(\omega(B)) - f(\omega(B'))) = (1 - A(x))\lambda \neq 0,$$

which has a constant sign. Therefore, we deduce $A_B(x) < A'_{B'}(x)$ or $A_B(x) > A'_{B'}(x)$ for any $x \in \text{supp}(A) \setminus \text{core}(A)$. This means that $A_B^\alpha \subset A'_{B'}^\alpha$ or $A'_{B'}^\alpha \subset A_B^\alpha$ where $\alpha = \max\{\beta \in (0, 1] \mid A_B^\beta \neq A'_{B'}^\beta\}$. Now, using that the total order $\preceq_{\Delta_{\delta_A}^\downarrow}$ on $\mathcal{A}_1^{L_n A}$ is generated from $\preceq_{\delta_A}$, which is an admissible order with respect to $\preceq_{\text{inc}}$ in the sense of Definition 14, we have that $A'_{B'}^\alpha \subset A_B^\alpha \iff A_B \prec_{\Delta_{\delta_A}^\downarrow} A'_{B'}$. Let us prove in further detail this last step.

$\Rightarrow$) If $A'_{B'}^\alpha \subset A_B^\alpha$, we know that $A_B^\alpha \prec_{\text{inc}} A'_{B'}^\alpha$ and since $\preceq_{\delta_A}$ is an admissible order with respect to $\preceq_{\text{inc}}$, we get $A_B^\alpha \prec_{\delta_A} A'_{B'}^\alpha$. Finally, taking into account that $\alpha = \max\{\beta \in (0, 1] \mid A_B^\beta \neq A'_{B'}^\beta\}$ and the total order $\preceq_{\Delta_{\delta_A}^\downarrow}$ on $\mathcal{A}_1^{L_n A}$ is generated from $\preceq_{\delta_A}$, then

$$A_B \prec_{\Delta_{\delta_A}^\downarrow} A'_{B'}.$$

$\Leftarrow$) Consider now $A_B \prec_{\Delta_{\delta_A}^\downarrow} A'_{B'}$. Recall that we had that either $A_B^\alpha \subset A'_{B'}^\alpha$ or $A'_{B'}^\alpha \subset A_B^\alpha$

where $\alpha = \max\{\beta \in (0, 1] \mid A_B^\beta \neq A'_{B'}^\beta\}$. If $A_B^\alpha \subset A'_{B'}^\alpha$, this implies that $A'_{B'}^\alpha \preceq_{\text{inc}} A_B^\alpha$, which implies $A'_{B'}^\alpha \prec_{\delta_A} A_B^\alpha$. Therefore, $A'_{B'} \prec_{\Delta_{\delta_A}^\downarrow} A_B$ and a contradiction arises. So, it must hold that $A'_{B'}^\alpha \subset A_B^\alpha$.

Finally, the result follows since

$$\begin{aligned} B \prec_{\Delta_{\delta_B}^\downarrow} B' &\iff \omega(B) < \omega(B') \iff A'_{B'}^\alpha \subset A_B^\alpha \iff A_B \prec_{\Delta_{\delta_A}^\downarrow} A'_{B'} \\ &\iff Z(A, B) \prec Z(A', B'). \end{aligned}$$

b) Suppose $B' = B$, therefore $\omega(B) = \omega(B')$. Let $A^\alpha, (A')^\alpha$ be the α -cuts where $\alpha = \max\{\beta \in (0, 1] \mid A^\beta \neq (A')^\beta\}$ and $A_B^{\alpha_B}, (A'_B)^{\alpha_B}$ be the α_B -cuts where $\alpha_B = \alpha + f(\omega(B))(1 - \alpha)$. By Lemma 3 we have that $\alpha_B = \max\{\beta \in (0, 1] \mid A_B^\beta \neq (A'_B)^\beta\}$. Moreover, by Theorem 2 it holds that $A^\alpha = A_B^{\alpha_B}$ and $(A')^\alpha = (A'_B)^{\alpha_B}$. Therefore, we have

$$A \preceq_{\Delta_{\delta_A}^\downarrow} A' \iff A^\alpha \preceq_{\delta_A} (A')^\alpha \iff A_B^{\alpha_B} \preceq_{\delta_A} (A'_B)^{\alpha_B} \iff Z(A, B) \preceq Z(A', B).$$

□

5 Practical example

Let us suppose that an NBA Team wants to analyze how good are the six best draft prospects of a given year. Hence, they decide to hire a professional scout that analyzes the six prospects $E = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ to determine the capability of each player on the court. This

Table 2 Labels of linguistic chains $\mathcal{L}_{n_A}$ and $\mathcal{L}_{n_B}$

$\mathcal{L}_{n_A}$	$\mathcal{L}_{n_B}$
N = No Eligible	I = Impossible
AN = Almost No Eligible	NS = Not Sure
VL = Very Low Level	M = Maybe
L = Low Level	P = Possibly
BA = Below Average	SS = Somewhat Sure
A = Average	S = Sure
AA = Above Average	AS = Absolutely Sure
H = High Level	
VH = Very High Level	
PL = Pro Level	
RS = Rising Star	

expert evaluates the variable $X = \{\text{Performance on the court}\}$ with some degree of confidence and for each of the six prospects, the scout makes an evaluation (using second component finite-valued Z -numbers) in order to determine which player would be the most appropriate to select to join the team.

We will denote by $Z(A_i, B_i)$ the evaluation of the scout on player P_i for all $1 \leq i \leq 6$. In our example, we assume

$$n_A = 10, n_B = 6, m = 6, Y_m = \{0, 0.2, 0.4, 0.6, 0.8, 1\},$$

and $\mathcal{L}_{n_A} = \{N, AN, VL, L, BA, A, AA, H, VH, PL, RS\}$ and $\mathcal{L}_{n_B} = \{I, NS, M, P, SS, S, AS\}$ are linguistic chains with the labels collected in Table 2. Using Proposition 1, there are $\binom{6+2 \cdot 6-2}{2 \cdot 6-2} = \binom{16}{10} = 8008$ dfns in the set $\mathcal{A}_1^{L_6 \times Y_6}$.

Accordingly to this notation, the evaluations provided by the scout for each of the players are the following:

- Player 1: The scout states the following evaluation with regard to Player 1:

“I am not sure that his performance on the court is high level.”

This sentence can be interpreted as the Z -valuation

$$Z(\text{Performance on the court, High Level, Not Sure}),$$

or in a simplified way $Z(\text{High Level, Not Sure})$, and, following the established notation and convention, the evaluation expressed as a couple of dfn's is:

$$Z(A_1, B_1) = (A_1 = \{0.6/AA, 1/H, 0.6/VH\}, B_1 = \{0.6/I, 1/NS, 0.8/M\}).$$

- Player 2: The scout states the following evaluation with regard to Player 2:

“I am absolutely sure that his performance on the court is below average.”

This sentence can be interpreted as the Z -valuation

$$Z(\text{Performance on the court, Below Average, Absolutely Sure}),$$

or in a simplified way $Z(\text{Below Average, Absolutely Sure})$, and, following the established notation and convention,

$$Z(A_2, B_2) = (A_2 = \{0.4/L, 1/BA, 0.4/A\}, B_2 = \{0.8/S, 1/AS\}).$$

- The scout states the following evaluation with regard to Player 3:

“I am not sure that his performance on the court is between below average and above average.”

This sentence can be interpreted as the Z -valuation

$Z(\text{Performance on the court, between Average and Above Average, Not Sure})$,
or in a simplified way $Z(\text{between Average and Above Average, Not Sure})$, and, following the established notation and convention,

$$Z(A_3, B_3) = (A_3 = \{0.4/BA, 1/A, 1/AA, 0.2/H\}, B_3 = \{0.6/I, 1/NS, 0.8/M\}).$$

- The scout states the following evaluation with regard to Player 4:

“My evaluation is that his performance on the court is high level but my certainty is between not sure and maybe.”

This sentence can be interpreted as the Z -valuation

$Z(\text{Performance on the court, High Level, between Not Sure and Maybe})$,
or in a simplified way $Z(\text{High Level, between Not Sure and Maybe})$, and, following the established notation and convention,

$$Z(A_4, B_4) = (A_4 = \{0.4/AA, 1/H, 0.4/VH\}, B_4 = \{0.4/I, 1/NS, 1/M, 0.8/P\}).$$

- The scout states the following evaluation with regard to Player 5:

“Possibly his performance on the court is between average and above average”

This sentence can be interpreted as the Z -valuation

$Z(\text{Performance on the court, between Average and Above Average, Possibly})$,
or in a simplified way $Z(\text{between Average and Above Average, Possibly})$, and, following the established notation and convention,

$$Z(A_5, B_5) = (A_5 = \{1/A, 1/AA, 0.2/H\}, B_5 = \{0.6/M, 1/P, 0.8/SS\}).$$

- The scout states the following evaluation with regard to Player 6:

“I am sure that his performance on the court is between below average and above average.”

This sentence can be interpreted as the Z -valuation

$Z(\text{Performance on the court, between Below Average and Above Average, Sure})$,
or in a simplified way $Z(\text{between Below Average and Above Average, Sure})$, and, following the established notation and convention,

$$Z(A_6, B_6) = (A_6 = \{1/BA, 1/A, 1/AA\}, B_6 = \{0.8/P, 0.8/SS, 1/S, 0.6/AS\}).$$

The first step is to compute the relative rank ω in $A_1^{L_6 \times Y_6}$ for all the second components B_i for all $1 \leq i \leq 6$ of the discrete Z -numbers.

With respect to Player 1, the position of dfn B_1 is 813. Then, the value of $\omega(B_1)$ is $\omega(B_1) = \frac{813}{8008} \approx 0.1015235$.

Table 3 collects these relative ranks along with the positions of dfns B_i where we have considered the total order $\preceq_{\Delta_6^1}$ on $A_1^{L_6}$ generated from the total order given in Definition 15.

Table 3 Relative ranks in $\mathcal{A}_1^{L_6 \times Y_6}$ of all dfns B_i

Z-number	B_i	Position of B_i	$w(B_i)$
$Z(A_1, B_1)$	$\{0.6/I, 1/NS, 0.8/M\}$	813	$\frac{813}{8008} \approx 0.1015235$
$Z(A_2, B_2)$	$\{0.8/S, 1/AS\}$	7924	$\frac{7924}{8008} \approx 0.9895105$
$Z(A_3, B_3)$	$\{0.6/I, 1/NS, 0.8/M\}$	813	$\frac{813}{8008} \approx 0.1015235$
$Z(A_4, B_4)$	$\{0.4/I, 1/NS, 1/M, 0.8/P\}$	1323	$\frac{1323}{8008} \approx 0.1652098$
$Z(A_5, B_5)$	$\{0.6/M, 1/P, 0.8/SS\}$	4602	$\frac{4602}{8008} \approx 0.5746753$
$Z(A_6, B_6)$	$\{0.8/P, 0.8/SS, 1/S, 0.6/AS\}$	7197	$\frac{7197}{8008} \approx 0.8987263$

The next step is to compute $(A_i)_{B_i}$ for all $1 \leq i \leq 6$, where the function $f(x) = 1 - x$ is chosen. With respect to Player 1, the value of the dfn $(A_1)_{B_1}$ is:

$$\begin{aligned}
 (A_1)_{B_1}(N) &= (A_1)_{B_1}(0) = 0, \\
 (A_1)_{B_1}(AN) &= (A_1)_{B_1}(1) = 0, \\
 (A_1)_{B_1}(VL) &= (A_1)_{B_1}(2) = 0, \\
 (A_1)_{B_1}(L) &= (A_1)_{B_1}(3) = 0, \\
 (A_1)_{B_1}(BA) &= (A_1)_{B_1}(4) = 0, \\
 (A_1)_{B_1}(A) &= (A_1)_{B_1}(5) = 0, \\
 (A_1)_{B_1}(AA) &= (A_1)_{B_1}(6) \\
 &= A_1(6) + f(w(B_1))(1 - A_1(6)) = 0.6 + f\left(\frac{813}{8008}\right)(1 - 0.6) \\
 &= 0.6 + \left(1 - \frac{813}{8008}\right) \cdot 0.4 \approx 0.6 + 0.8984765 \cdot 0.4 = 0.9593906, \\
 (A_1)_{B_1}(H) &= (A_1)_{B_1}(7) = 1 + f\left(\frac{813}{8008}\right)(1 - 1) = 1, \\
 (A_1)_{B_1}(VH) &= (A_1)_{B_1}(8) = 0.6 + f\left(\frac{813}{8008}\right)(1 - 0.6) = 0.9593906, \\
 (A_1)_{B_1}(PL) &= (A_1)_{B_1}(9) = 0, \\
 (A_1)_{B_1}(RS) &= (A_1)_{B_1}(10) = 0.
 \end{aligned}$$

Consequently, the dfn $(A_1)_{B_1}$ will be:

$$(A_1)_{B_1} = \{0.9594/AA, 1/H, 0.9594/VH\}.$$

The expressions of all dfns $(A_i)_{B_i}$, $i = 1, \dots, 6$ are shown in Table 4.

Finally, applying the total order $\leq$ defined in Definition 12 where we have considered the total orders $\leq_{\Delta_\delta^\downarrow}$ (generated from the interval total order $\leq_{t-inc}$ given in Definition 15) as $\leq_A$ and $\leq_{\Delta_\delta^\downarrow}$ (generated from the interval total order lex1) as $\leq_B$, the players are sorted according to the following rank of evaluations:

$$Z(A_2, B_2) \prec Z(A_6, B_6) \prec Z(A_3, B_3) \prec Z(A_5, B_5) \prec Z(A_1, B_1) \prec Z(A_4, B_4).$$

Table 4 Expressions of the dfns $(A_i)_{B_i}$, $i = 1, \dots, 6$

i	A_i	$\omega(B_i)$	$(A_i)_{B_i}$
1	{0.6/AA, 1/H, 0.6/VH}	0.1015	{0.9594/AA, 1/H, 0.9594/VH}
2	{0.4/L, 1/BA, 0.4/A}	0.9895	{0.4063/L, 1/BA, 0.4063/A}
3	{0.4/BA, 1/A, 1/AA, 0.2/H}	0.1015	{0.9391/BA, 1/A, 1/AA, 0.9188/H}
4	{0.4/AA, 1/H, 0.4/VH}	0.1652	{0.9009/AA, 1/H, 0.9009/VH}
5	{1/A, 1/AA, 0.2/H}	0.5747	{1/A, 1/AA, 0.5403/H}
6	{1/BA, 1/A, 1/AA}	0.8987	{1/BA, 1/A, 1/AA}

6 Comparison with other methods in the literature

The study of ranking methods, as mentioned in the introduction, is a subject in constant evolution. From the analysis of the different ranking methods of Z -numbers that have been published in the literature, a comparison among these ranking methods is mathematically problematic. Indeed,

- the probabilistic approach that is inherent to most of the Z -numbers approaches and which is not present in our approach,
- the typology of the components of the Z -numbers that depending on the approach can be triangular or trapezoidal fuzzy (generalised) numbers and not discrete fuzzy numbers as in our approach,
- the different ordering methodologies used,

jeopardize the conclusions derived from any comparison study. Taking into account the above discussion, after a thorough analysis of the literature, we have only been able to find two proposals that will allow us to make a comparison according to the interpretation of the Z -numbers proposed in this work. First, in Bilgin and Alci (2022) an extensive study of ranking techniques for Z -numbers is performed and specifically, there is a section devoted to the particular case of discrete Z -numbers, where Gong's method (Gong et al. 2020) is presented. Second, Xian et al. in (2021) proposed a method to sort Z -mixture numbers, and particularly discrete Z -numbers. These two approaches will be analyzed and compared with our ranking method. Anyway, it is important to highlight that the authors of both papers did not study whether these orders are total, a key property which has been proved for the order presented in our paper.

The discrete Z -numbers we will use in our comparison are shown in Table 5 and coincide with those used in the previous section. According to the sorting method proposed in this paper, the following results were obtained

$$Z(A_2, B_2) < Z(A_6, B_6) < Z(A_3, B_3) < Z(A_5, B_5) < Z(A_1, B_1) < Z(A_4, B_4).$$

Note that our method allow us to rank Z -numbers whose components are not necessary discrete triangular fuzzy numbers (discrete fuzzy numbers whose core contains a unique value).

In Xian et al. (2021), proposed an order for Z -mixture-numbers based on the positive and negative ideal degree. In that paper, given two Z -numbers Z_1 and Z_2 , the ideal degree is defined based on a parameter $\beta \in [0, 1]$ that balances the influence of each Z -number, see Xian et al. (2021) for details. All the examples considered in that paper are Z -numbers whose components are discrete triangular fuzzy numbers. When we apply the Xian's method

Table 5 Expressions of the $Z_i(A_i, B_i)$, $i = 1, \dots, 6$

$Z_i(A_i, B_i)$	A_i	B_i
$Z_1(A_1, B_1)$	{0.6/6, 1/7, 0.6/8}	{0.6/0, 1/1, 0.8/2}
$Z_2(A_2, B_2)$	{0.4/3, 1/4, 0.4/5}	{0.8/5, 1/6}
$Z_3(A_3, B_3)$	{0.4/4, 1/5, 1/6, 0.2/7}	{0.6/0, 1/1, 0.8/2}
$Z_4(A_4, B_4)$	{0.4/6, 1/7, 0.4/8}	{0.4/0, 1/1, 1/2, 0.8/3}
$Z_5(A_5, B_5)$	{1/5, 1/6, 0.2/7}	{0.6/2, 1/3, 0.8/4}
$Z_6(A_6, B_6)$	{1/4, 1/5, 1/6}	{0.8/3, 0.8/4, 1/5, 0.6/6}

to order the discrete Z -numbers considered in Table 5, we find that the discrete Z -numbers $Z(A_2, B_2)$ and $Z(A_6, B_6)$ cannot be considered because the computation of the centroid points associated with components B_2 and A_6 was impossible to perform. This is because B_2 has only two values on the support and, in the case of A_6 , it is described only by the core which contains three values.

With the other discrete Z -numbers, we have obtained the following result depending on the parameter β :

$$\begin{aligned}\beta \in [0, 0.25] : Z(A_3, B_3) < Z(A_4, B_4) < Z(A_5, B_5) < Z(A_1, B_1), \\ \beta \in [0.25, 1] : Z(A_3, B_3) < Z(A_4, B_4) < Z(A_1, B_1) < Z(A_5, B_5).\end{aligned}$$

Therefore, the only discordant Z -number is $Z(A_4, B_4)$, which is placed in third place while according to our sorting order, it is in first place. We also observe that parameter β plays an important role as depending on its value, the Z -number $Z(A_5, B_5)$ is placed in first place. As a conclusion we obtain that our procedure does not present any problem when choosing any discrete fuzzy number as a component of the corresponding discrete Z -number, a problem that Xian's method has.

In Gong et al. (2020), suggest a ranking method based on transforming each discrete Z -number into a discrete fuzzy subset. Once its corresponding discrete fuzzy subset has been obtained for each discrete Z -number, the latter is ordered according to the method proposed in Basirzadeh et al. (2012). For the transformation of a discrete Z -number, the degree of fuzziness of the second component Kang et al. (2019) is used (for more details see Gong et al. (2020)). In the case of discrete Z -number $Z_4 = (A_4, B_4)$, the method proposed by Gong cannot be applied. This is because only discrete Z -numbers whose second components have a unique value as their core are allowed. Note that component B_4 has as its core the interval $[1, 2]$.

With the other discrete Z -numbers, we have obtained the following ordering when we take the parameter $\alpha = 0$ (see Gong et al. (2020) for more details about the parameter α).

$$Z(A_5, B_5) < Z(A_6, B_6) < Z(A_2, B_2) < Z(A_1, B_1) < Z(A_3, B_3).$$

From the comparison of the two orderings, it can be deduced, from the point of view of linguistic interpretation, that it does not make much sense that $Z_1 = (A_1, B_1) < Z_3 = (A_3, B_3)$ is obtained by applying Gong's method. Note that the second components are the same. Furthermore, if we compare the first components, we see that the linguistic assessment described by A_1 is intuitively more favorable than A_3 . Finally, if we apply any of the total orders described in Riera et al. (2021) we have that $A_3 < A_1$, a fact that coincides with our initial intuition.

In summary, from the analysis of the compared methods it is clear that the ranking method proposed in this paper allows us to order discrete Z -numbers whose components do not

necessarily have a unique element as their core, a problem that we have seen present in both cases. Furthermore, the obtained results are consistent with linguistic intuition.

7 Conclusions

Zadeh's Z -numbers have positioned themselves as an interesting computational linguistic model that has been used in multiple applications, and in particular, in decision-making problems. In this field, it is essential to have adequate procedures, consistent with the linguistic model, that allow ordering experts' preferences/opinions expressed as Z -valuations. Thus, in this paper we carry out for the first time a study on total orders in the set of discrete Z -numbers (and more specifically, for the subset of second component finite-valued discrete Z -numbers $\mathcal{A}_1^{L_n \times Y_m}$). The order is based on a transformation operator in the set of discrete fuzzy numbers that changes the uncertainty of the first component of a discrete Z -number (increasing the membership values outside of the core) according to the reliability/confidence of this first component provided by the second component of the discrete Z -number. Moreover, this order has a clear linguistic interpretation that is intended to align with human intuition when some of the components of the Z -numbers are equal and a suitable interval order is chosen. For this purpose, the cardinal of the set $\mathcal{A}_1^{L_n \times Y_m}$ has been computed and the construction of a relative order function (constructed from the total order defined in $\mathcal{A}_1^{L_n}$) compatible with this order has been proposed. To ease the computation of the order, we include several algorithms and specifically, an algorithm that allows to generate and order all the dfn of $\mathcal{A}_1^{L_n}$ based on this order. Also, we have provided an illustrative example that shows how the computation of the order works in a practical situation. Finally, a comparative section has been included with the intention of analyzing the validity of our method compared to others that exist in the literature.

In future work, we plan to define aggregation functions for discrete Z -numbers based on the total order we have established. Additionally, we intend to expand this order for the so-called mixed Z -numbers $Z(A, B)$, in which A is not necessarily a dfn but it can be any fuzzy number.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

- Aliev RA, Alizadeh AV, Huseynov OH (2015) The arithmetic of discrete Z-numbers. *Inf Sci* 290:134–155. <https://doi.org/10.1016/j.ins.2014.08.024>
- Bailey WN (1964) Generalized hypergeometric series. In: Bailey WN Cambridge tracts in mathematics and mathematical physics, <https://books.google.es/books?id=TVyswgEACAAJ>
- Bakar A, Gegov A (2015) Multi-layer decision methodology for ranking Z-numbers. *Int J Comput Intell Syst* 8(2):395–406
- Banerjee R, Pal SK, Pal JK (2022) A decade of the Z-numbers. *IEEE Trans Fuzzy Syst* 30(8):2800–2812. <https://doi.org/10.1109/TFUZZ.2021.3094657>
- Basirzadeh H, Farnam M, Hakimi E (2012) An approach for ranking discrete fuzzy sets. *J Math Comput* 2(3):583–592
- Biggs N (2002) Discrete Mathematics. Oxford University Press, UK
- Bilgin F, Alci M (2022) A review on ranking of Z-numbers. *J Comput Sci Res* 4(2):1–12
- Bustince H, Fernandez J, Kolesárová A, Mesiar R (2013) Generation of linear orders for intervals by means of aggregation functions. *Fuzzy Sets Syst* 220:69–77. <https://doi.org/10.1016/j.fss.2012.07.015>
- Casasnovas J, Riera JV (2011) Extension of discrete t-norms and t-conorms to discrete fuzzy numbers. *Fuzzy Sets Syst* 167(1):65–81
- Chai J, Su Y, Lu S (2023) Linguistic z-number preference relation for group decision making and its application in digital transformation assessment of smes. *Expert Syst Appl* 213:118749
- Cheng R, Zhang J, Kang B (2022) Ranking of Z-numbers based on the developed golden rule representative value. *IEEE Trans Fuzzy Syst* 30(12):5196–5210. <https://doi.org/10.1109/TFUZZ.2022.3170208>
- Chiclana F, Agell N, Wu J, Herrera-Viedma E (2015) Fuzzy decision-making and consensus: new trends and real-life applications. *Appl Soft Comput* 35:789–791. <https://doi.org/10.1016/j.asoc.2015.07.043>
- Chutia R (2021) Ranking of Z-numbers based on value and ambiguity at levels of decision making. *Int J Intell Syst* 36:313–331
- De Miguel L, Bustince H, Fernandez J, Induráin E, Kolesárová A, Mesiar R (2016) Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making. *Inf Fusion* 27:189–197. <https://doi.org/10.1016/j.inffus.2015.03.004>
- Doukas H, Karakosta C, Psarras J (2010) Computing with words to assess the sustainability of renewable energy options. *Expert Syst Appl* 37(7):5491–5497
- Ezadi S, Allahviranloo T (2017) New multi-layer method for Z-number ranking using hyperbolic tangent function and convex combination. *Intell Autom Soft Comput* 24:1–7. <https://doi.org/10.1080/10798587.2017.1367146>
- Ezadi S, Allahviranloo T, Mohammadi S (2018) Two new methods for ranking of Z-numbers based on sigmoid function and sign method. *Int J Intell Syst* 33:1476–1487
- Gong Y, Li X, Jiang W (2020) A new method for ranking discrete Z-number. In: 2020 Chinese Control And Decision Conference (CCDC), pp 3591–3596
- Herrera F, Herrera-Viedma E (2000) Linguistic decision analysis: steps for solving decision problems under linguistic information. *Fuzzy Sets Syst* 115(1):67–82. [https://doi.org/10.1016/S0165-0114\(99\)00024-X](https://doi.org/10.1016/S0165-0114(99)00024-X)
- Herrera F, Alonso S, Chiclana F, Herrera-Viedma E (2009) Computing with words in decision making: foundations, trends and prospects. *Fuzzy Optim Decis Making* 8(5):43–80. <https://doi.org/10.1007/s10700-009-9065-2>
- Jiang W, Xie C, Luo Y, Tang Y (2017) Ranking Z-numbers with an improved ranking method for generalized fuzzy numbers. *J Intell Fuzzy Syst* 32(3):1931–1943
- Kang B, Wei D, Li Y, Deng Y (2012) A method of converting Z-number to classical fuzzy number. *J Inf Comput Sci* 9(3):703–709
- Kang B, Deng Y, Hewage K, Sadiq R (2019) A method of measuring uncertainty for Z-number. *IEEE Trans Fuzzy Syst* 27(4):731–738. <https://doi.org/10.1109/TFUZZ.2018.2868496>
- Li C, Dong Y, Herrera F, Herrera-Viedma E, Martínez-López L (2017) Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. *Inf Fusion* 33:29–40. <https://doi.org/10.1016/j.inffus.2016.04.005>
- Lima A, Palmeira ES, Bedregal B, Bustince H (2021) Multidimensional fuzzy sets. *IEEE Trans Fuzzy Syst* 29(8):2195–2208. <https://doi.org/10.1109/TFUZZ.2020.2994997>
- Massanet S, Riera JV, Torrens J, Herrera-Viedma E (2014) A new linguistic computational model based on discrete fuzzy numbers for computing with words. *Inf Sci* 258:277–290. <https://doi.org/10.1016/j.ins.2013.06.055>
- Massanet S, Riera JV, Torrens J (2016) A new vision of Zadeh's Z-numbers. In: Carvalho JP, Lesot M-J, Kaymak U, Vieira S, Bouchon-Meunier B, Yager RR (eds) Information processing and management of uncertainty in knowledge-based systems. Springer, Cham, pp 581–592

- Massanet S, Riera JV, Torrens J (2020) A new approach to Zadeh's Z-numbers: mixed-discrete Z-numbers. *Inf Fusion* 53:35–42. <https://doi.org/10.1016/j.inffus.2019.06.015>
- Morente-Molinera JA, Pérez IJ, Ureña MR, Herrera-Viedma E (2015) On multi-granular fuzzy linguistic modeling in group decision making problems: A systematic review and future trends. *Knowl-Based Syst* 74:49–60. <https://doi.org/10.1016/j.knosys.2014.11.001>
- Munar M, Massanet S, Ruiz-Aguilera D (2023) On the cardinality of some families of discrete connectives. *Inf Sci* 621:708–728. <https://doi.org/10.1016/j.ins.2022.10.121>
- Nazari-Shirkouhi S, Tavakoli M, Govindan K, Mousakhani S (2023) A hybrid approach using z-number dea model and artificial neural network for resilient supplier selection. *Expert Syst Appl* 222:119746
- Pal SK, Banerjee R, Dutta S, Sarma S (2013) An insight into the Z-number approach to CWW. *Fund Inf* 124:197–229
- Petkovsek M, Wilf HS, Zeilberger D (1996) $A = B$. Taylor and Francis group, UK
- Riera JV, Torrens J (2012) Aggregation of subjective evaluations based on discrete fuzzy numbers. *Fuzzy Sets Syst* 191:21–40
- Riera JV, Torrens J (2014) Aggregation functions on the set of discrete fuzzy numbers defined from a pair of discrete aggregations. *Fuzzy Sets Syst* 241:76–93
- Riera JV, Massanet S, Herrera-Viedma E, Torrens J (2015) Some interesting properties of the fuzzy linguistic model based on discrete fuzzy numbers to manage hesitant fuzzy linguistic information. *Appl Soft Comput* 36:383–391. <https://doi.org/10.1016/j.asoc.2015.07.022>
- Riera JV, Massanet S, Bustince H, Fernandez J (2021) On admissible orders on the set of discrete fuzzy numbers for application in decision making problems. *Mathematics* 9(1):95. <https://doi.org/10.3390/math9010095>
- Santana F, Bedregal B, Viana P, Bustince H (2020) On admissible orders over closed subintervals of $[0,1]$. *Fuzzy Sets Syst* 399:44–54. <https://doi.org/10.1016/j.fss.2020.02.009>
- Voxman W (2001) Canonical representations of discrete fuzzy numbers. *Fuzzy Sets Syst* 118(3):457–466. [https://doi.org/10.1016/S0165-0114\(99\)00053-6](https://doi.org/10.1016/S0165-0114(99)00053-6)
- Xian S, Chai J, Li T, Huang J (2021) A ranking model of Z-mixture-numbers based on the ideal degree and its application in multi-attribute decision making. *Inf Sci* 550:145–165. <https://doi.org/10.1016/j.ins.2020.10.038>
- Xu Z, Yager R (2006) Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int J General Syst* 35(4):417–433
- Yager RR (2016) Golden rule representative values for non-standard membership grades. In: 2016 IEEE 8th International Conference on Intelligent Systems (IS), pp 2–7
- Yager R (2012) On Z-valuations using Zadeh's Z-numbers. *Int J Intell Syst* 27:259–278
- Yuan J, Lu W, Ding H, Liu J, Mahmoudi A (2023) A novel z-number based real option (zro) model under uncertainty: application in public-private-partnership refinancing value evaluation. *Expert Syst Appl* 213:118808
- Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning-III. *Inf Sci* 9(1):43–80. [https://doi.org/10.1016/0020-0255\(75\)90017-1](https://doi.org/10.1016/0020-0255(75)90017-1)
- Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning-I. *Inf Sci* 8(3):199–249. [https://doi.org/10.1016/0020-0255\(75\)90036-5](https://doi.org/10.1016/0020-0255(75)90036-5)
- Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning-II. *Inf Sci* 8(4):301–357. [https://doi.org/10.1016/0020-0255\(75\)90046-8](https://doi.org/10.1016/0020-0255(75)90046-8)
- Zadeh LA (2011) A note on Z-numbers. *Inf Sci* 181(14):2923–2932. <https://doi.org/10.1016/j.ins.2011.02.022>
- Zuheros C, Li C, Cabrerizo FJ, Dong Y, Herrera-Viedma E, Herrera F (2018) Computing with words: revisiting the qualitative scale. *Int J Uncertain Fuzziness Knowl-Based Syst* 26(Supplement–2):127–143. <https://doi.org/10.1142/S0218488518400147>