Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 451 (2022) 176–195

FUZZY sets and systems

www.elsevier.com/locate/fss

Fuzzy implication functions with a specific expression: The polynomial case

Sebastia Massanet [a,b,*], Arnau Mir [a,b], Juan Vicente Riera [a,b], Daniel Ruiz-Aguilera [a,b]

[a] *Department of Mathematics and Computer Science, University of the Balearic Islands, Cra. de Valldemossa, km 7.5, 07122 Palma, Spain*
[b] *Health Research Institute of the Balearic Islands (IdISBa), 07010 Palma, Spain*

Received 15 January 2022; received in revised form 6 April 2022; accepted 18 June 2022
Available online 23 June 2022

Abstract

In the last decades, more than a hundred families of fuzzy implication functions have been proposed. These families are generated by adequately combining other logical connectives or univariate functions. However, no attention has been given to the final expression of the fuzzy implication function, a crucial feature for any application. In this paper, fuzzy polynomial implications are introduced as those fuzzy implication functions whose expression is given by a polynomial of two variables. Polynomials present advantages with respect to other types of functions making them interesting for applications. Several results are proved for polynomials of any degree and the characterisation of all fuzzy polynomial implications of degree less or equal to 4 is achieved.

Keywords: Fuzzy implication functions; Polynomials; Monotonicities

1. Introduction

Up to now, more than a hundred families of fuzzy implication functions have been introduced in the literature (see [3,4,8] and references therein). From all these studies, experts are able to choose the fuzzy implication function which satisfies all the additional properties needed for a specific application. However, there is an important aspect for any application that has been neglected: the final expression of the fuzzy implication function.

Indeed, the vast majority of the families of fuzzy implication functions have been proposed through either (i) an adequate combination of logical connectives (fuzzy negations, fuzzy conjunctions, fuzzy disjunctions or other fuzzy implication functions) or (ii) by using univariate functions. While (S,N), R or QL-implications are important examples in the first category, Yager's f and g-generated implications are the prototypes in the second one. Although members of any family share a common way to be generated, their expressions can be completely different, depending mainly on the expressions of the components used to generate them. For a matter of visualisation, consider the Yager's

---

[*] Corresponding author at: Department of Mathematics and Computer Science, University of the Balearic Islands, Cra. de Valldemossa, km 7.5, 07122 Palma, Spain.

E-mail addresses: s.massanet@uib.es (S. Massanet), arnau.mir@uib.es (A. Mir), jvicente.riera@uib.es (J.V. Riera), daniel.ruiz@uib.es (D. Ruiz-Aguilera).

https://doi.org/10.1016/j.fss.2022.06.016
0165-0114/

f-implications generated from $f_1(x) = 1 - x$ and $f_2(x) = -\ln\left(\frac{3^x-1}{2}\right)$ for all $x \in [0, 1]$. These fuzzy implication functions belong to the same family but their final expressions are quite different:

$$I_{f_1}(x, y) = 1 - x + xy, \quad I_{f_2}(x, y) = \log_3 (1 + 2^{1-x}(3^y - 1)^x), \quad x, y \in [0, 1].$$

This phenomenon could be overlooked if the final expression of the fuzzy implication function did not play any role when it is applied. However, it is obvious that in addition to the additional properties the fuzzy implication function may fulfil, the final expression of the operator determines greatly the behaviour of the application and even it can be the differential factor in order to choose one fuzzy implication function over the others. Thus, the main motivation of this work is to start with a deep study of fuzzy implication functions in terms of their final expression and not in terms of their construction method or their additional properties. Some preliminary results without proofs on this topic have been presented in several conferences [9–11].

In this first paper, the goal is to study fuzzy implication functions whose expression is a polynomial of two variables. Polynomial functions have been always one of the most studied and used types of functions. Several features support their popularity. First of all, polynomials present rather simple expressions, easily to be implemented in a computer, which is a crucial characteristic when choosing a type of function to model an operator in an specific application. Moreover, they are always continuous and, in comparison with other families of functions such as exponential or rational functions, they do not propagate as much error from inputs to outputs.

It is important to note that in other contexts such as Archimedean continuous t-norms and uninorms, there exist similar studies [1,7] but for rational operators directly without a first study on polynomials. This is a direct consequence of the axioms of t-norms and uninorms with respect to the ones of fuzzy implication functions. While t-norms and uninorms are associative (which is a very restrictive property), fuzzy implication functions require only the monotonicities and some boundary conditions, leading to a greater number of fuzzy implication functions whose expression is a polynomial in comparison with t-norms and uninorms.

The paper is structured as follows. After recalling the basic definitions and results which will be used throughout the paper, in Section 3 the definition of a fuzzy polynomial implication of any degree is presented and some additional properties are studied. Moreover, some conditions on the coefficients are determined to ensure the fulfilment of the boundary conditions of the definition of a fuzzy implication function. However, the impossibility to characterise in general which polynomials of two variables are fuzzy implication function leads to the study of polynomials of low degree in Section 4. Indeed, complete characterisations of all fuzzy polynomial implications of degree less or equal to 4 are proved. As it has been already said, some preliminary results were presented in [9] without proofs. In this paper, some of the results presented in the conference paper are corrected and the case of polynomials of degree 4 is addressed (and solved) for the first time. The paper ends with some conclusions and future work.

2. Preliminaries

Let us recall some concepts and results that will be used throughout this paper. First, we give the definition of fuzzy negation.

Definition 1 *([6, Definition 1.1]).* A decreasing function $N : [0, 1] \to [0, 1]$ is a *fuzzy negation* if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is

(i) *strict*, if it is continuous and strictly decreasing.
(ii) *strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0, 1]$.

A well-known fuzzy negation is the classical negation $N_{\mathbf{C}}(x) = 1 - x$ for all $x \in [0, 1]$, which is strong and therefore, also strict.

Next, we recall the definition of fuzzy implication functions.

Definition 2 *([6, Definition 1.15]).* A binary operator $I : [0, 1]^2 \to [0, 1]$ is called a *fuzzy implication function*, if it satisfies:

(I1) $I(x, z) \geq I(y, z)$ when $x \leq y$, for all $z \in [0, 1]$.

(I2) $I(x, y) \leq I(x, z)$ when $y \leq z$, for all $x \in [0, 1]$.
(I3) $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

From the definition, we can deduce that $I(0, x) = 1$ and $I(x, 1) = 1$ for all $x \in [0, 1]$ while the symmetrical values $I(x, 0)$ and $I(1, x)$ are not determined. Some additional properties of fuzzy implication functions which will be used in this work are:

- The *left neutrality principle*,

$$I(1, y) = y, \quad y \in [0, 1]. \tag{NP}$$

- The *exchange principle*,

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]. \tag{EP}$$

- The *law of importation* with respect to a t-norm T,

$$I(T(x, y), z) = I(x, I(y, z)), \quad x, y, z \in [0, 1]. \tag{LI}$$

- The *ordering property*,

$$x \leq y \Longleftrightarrow I(x, y) = 1, \quad x, y \in [0, 1]. \tag{OP}$$

- The *identity principle*,

$$I(x, x) = 1, \quad x \in [0, 1]. \tag{IP}$$

Finally, we recall the definitions of two important families of these operators, namely, (S,N)-implications and Yager's f-generated implications.

Definition 3 *([3, Definition 2.4.1]).* A function $I : [0, 1]^2 \to [0, 1]$ is called an *(S,N)-implication* if there exist a t-conorm S and a fuzzy negation N such that I is given by

$$I_{S,N}(x, y) = S(N(x), y), \quad x, y \in [0, 1].$$

Definition 4 *([3, Definition 3.1.1]).* Let $f : [0, 1] \to [0, +\infty]$ be a continuous and strictly decreasing function with $f(1) = 0$. The function $I_f : [0, 1]^2 \to [0, 1]$ defined by

$$I_f(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1],$$

understanding $0 \cdot (+\infty) = 0$, is called an *f-generated implication*.

3. Fuzzy polynomial implications: general properties

In this section, we will introduce the concept of fuzzy polynomial implication and we will prove some conditions on the coefficients of the polynomial in order to obtain a fuzzy implication function from this expression.

3.1. Definition and first additional properties

First of all, let us define these operators.

Definition 5. Let $n \in \mathbb{N}$. A binary operator $I : [0, 1]^2 \to [0, 1]$ is called a *fuzzy polynomial implication of degree n* if it is a fuzzy implication function and its expression is given by

$$I(x, y) = \sum_{\substack{0 \leq i, j \leq n \\ i+j \leq n}} a_{ij} x^i y^j$$

for all $x, y \in [0, 1]$, $a_{ij} \in \mathbb{R}$ and there exist some $0 \leq i, j \leq n$ with $i + j = n$ such that $a_{ij} \neq 0$.

The first question to answer is whether there exist fuzzy polynomial implications. The answer is clearly positive. For instance, the well-known Reichenbach implication given by $I_{\mathbf{RC}}(x, y) = 1 - x + xy$ for all $x, y \in [0, 1]$ is a fuzzy polynomial implication of degree 2. Furthermore, as the following example shows, there exist fuzzy polynomial implications of any degree $n \in \mathbb{N}$ with $n \geq 2$.

Example 1. Let us consider the parametrized family of fuzzy negations given by $N_n(x) = 1 - x^n$ for all $x \in [0, 1]$ and $n \in \mathbb{Z}^+$, and the probabilistic sum t-conorm, whose expression is $S_{\mathbf{P}}(x, y) = x + y - xy$ for all $x, y \in [0, 1]$. This t-conorm belongs to the family of Hamacher t-conorms (the dual t-conorms of the Hamacher t-norms given in [7]) and it is the unique t-conorm whose expression is given by a polynomial of two variables. Then, if we consider these two operators, we can construct the following parametrized family of (S,N)-implications

$$I_{S_{\mathbf{P}}, N_{n-1}}(x, y) = S_P(N_{n-1}(x), y) = 1 - x^{n-1} + x^{n-1}y$$

for all $x, y \in [0, 1]$ and $n \geq 2$. As it can be observed, they are polynomial implications of degree n. In addition, they satisfy ***(LI)*** with respect to the product t-norm, given by $T_{\mathbf{P}}(x, y) = xy$ for all $x, y \in [0, 1]$, and therefore, they are also Yager's f-generated implications with $f(x) = \sqrt[n-1]{1 - x}$ (see [12, Theorem 6]).

It is important to understand the concept of fuzzy polynomial implication correctly. Definition 5 does not include those fuzzy implication functions whose expression is piecewise polynomial such as the Łukasiewicz $I_{\mathbf{LK}}$ or Gödel $I_{\mathbf{GD}}$ implications given by

$$I_{\mathbf{GD}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y, \end{cases} \qquad I_{\mathbf{LK}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x + y & \text{if } x > y, \end{cases}$$

for all $x, y \in [0, 1]$, respectively.

Besides having a simple and pleasant expression, fuzzy polynomial implications satisfy also some additional properties with great importance from the point of view of applications. A first property which can be straightforwardly derived from the definition is the continuity.

Proposition 1. *All fuzzy polynomial implications are continuous.*

Proof. Since all polynomials are continuous in $\mathbb{R}^2$, they are also continuous in $[0, 1]^2$. □

Moreover, they have a trivial one region as the following result proves.

Proposition 2. *All fuzzy polynomial implications I have a trivial one region, i.e., $I(x, y) = 1$ if, and only if, $x = 0$ or $y = 1$. Consequently, they do not satisfy either* **(IP)** *or* **(OP)**.

Proof. Suppose, on the contrary, that there exist some $x_0 > 0$ and some $y_0 < 1$ such that $I(x_0, y_0) = 1$. Due to the monotonicities of a fuzzy implication function, $I(x, y) = 1$ for all $(x, y) \in [0, x_0] \times [y_0, 1]$. Since the unique polynomials which have a constant region are polynomials of degree 0, then $I(x, y) = 1$ for all $x, y \in [0, 1]$. However, this is a contradiction with **(I3)** because $I(1, 0) = 0 \neq 1$. Thus, I must have a trivial one region and consequently, it cannot satisfy either **(IP)** or **(OP)**. □

In [5] it is proved that having a trivial one region, i.e., $I(x, y) = 1$ if, and only if, $x = 0$ or $y = 1$, is essential to generate strong equality indices by using fuzzy implication functions.

Other additional properties need further insights about the feasible coefficients of the fuzzy polynomial implications in order to be analysed. However, next result links those fuzzy polynomial implications satisfying **(EP)** with the family of (S,N)-implications.

Proposition 3. *Let* $I(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$ *be a fuzzy polynomial implication of degree* n. *If* I *satisfies* **(EP)**, *then* I *is an (S,N)-implication generated by the strict fuzzy negation* $N(x) = \sum_{i=0}^{n} a_{i0} x^i$ *and the t-conorm* $S(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} (N^{-1}(x))^i y^j$.

Proof. First, since I is a fuzzy implication function, it satisfies **(I1)**. Moreover, its natural negation, which is given by $N_I(x) = I(x,0) = \sum_{i=0}^{n} a_{i0} x^i$ for all $x \in [0,1]$ is a strict negation. Thus, if I satisfies **(EP)**, by using Theorem 5.2 in [2], I is an (S,N)-implication generated from the fuzzy negation N_I and the t-conorm S given by

$$S(x,y) = I(N_I^{-1}(x), y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} (N^{-1}(x))^i y^j$$

for all $x, y \in [0,1]$. □

3.2. *On the choice of the coefficients*

Although Definition 5 provides a simple expression for a fuzzy implication function, it is clear that not all polynomials lead to a fuzzy implication function in the sense of Definition 2. As an example, consider the bivariate polynomial of degree 2 given by $p(x,y) = x^2$ which do not fulfil either **(I1)** or **(I3)**. Therefore, a characterisation of which coefficients $a_{ij} \in \mathbb{R}$ generate a polynomial $p(x,y)$ which satisfies the conditions of Definition 2 is necessary. The general problem is unfeasible due to its complexity. However, in this section, we will prove some necessary conditions for the coefficients of a polynomial of a general degree in order to obtain a fuzzy polynomial implication. Later on, in Section 4, a complete characterisation will be obtained for all polynomials of degree smaller or equal to 4.

First of all, the next result determines the necessary and sufficient conditions a polynomial must satisfy in order to be the expression of a fuzzy polynomial implication.

Theorem 1. *A bivariate polynomial of degree* n *given by* $p(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$ *for all* $x, y \in [0,1]$ *is a fuzzy polynomial implication if, and only if, the following properties hold:*

- *(i)* $p(0,y) = p(x,1) = 1$ *for all* $x, y \in [0,1]$.
- *(ii)* $p(1,0) = 0$.
- *(iii)* $\frac{\partial p(x,y)}{\partial x} \le 0$ *for all* $x, y \in [0,1]$.
- *(iv)* $\frac{\partial p(x,y)}{\partial y} \ge 0$ *for all* $x, y \in [0,1]$.

Proof. If $p(x,y)$ is a fuzzy polynomial implication, Properties (i)-(iv) follow directly from Definition 2.

Conversely, consider a polynomial satisfying Properties (i)-(iv). First of all, we want to ensure that $p(x,y)$ takes values in $[0,1]$ when $x, y \in [0,1]$. Indeed, using Properties (ii), (iii), (iv) and (i), we obtain:

$$0 = p(1,0) \le p(x,0) \le p(x,y) \le p(x,1) = 1 \text{ for all } x, y \in [0,1].$$

Moreover, Properties **(I1)**-**(I3)** from Definition 2 are also satisfied. Property **(I3)** is immediate from (i) and (ii). Properties **(I1)** and **(I2)** follow from Properties (iii) and (iv). □

Note that the previous theorem provides general conditions on the polynomial to be a fuzzy implication function, but it is necessary to study the consequences of these conditions on the coefficients of the polynomial. Properties (i) and (ii) can be easily managed as the following result shows.

Proposition 4. *Let* $p(x, y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$ *for all* $x, y \in [0, 1]$ *be a bivariate polynomial of degree* n. *Then the following statements hold:*

(i) $p(0, y) = 1$ *for all* $y \in [0, 1]$ *if, and only if,* $a_{00} = 1$ *and* $a_{0j} = 0$ *for all* $0 < j \le n$.

(ii) $p(x, 1) = 1$ *for all* $x \in [0, 1]$ *if, and only if,* $\sum_{j=0}^{n} a_{0j} = 1$ *and* $\sum_{j=0}^{n-i} a_{ij} = 0$ *for all* $0 < i \le n$.

(iii) $p(1, 0) = 0$ *if, and only if,* $\sum_{i=0}^{n} a_{i0} = 0$.

Proof. The results are straightforward. □

However, up to our knowledge, Properties (iii) and (iv) of Theorem 1 (those related with the derivatives) cannot be rewritten in terms of the coefficients of the polynomial for polynomials of a general degree. In the next section, it will be proved that for polynomials of degree smaller or equal to 4 this problem can be solved. The key result for the next section is the following result which provides a nice simplification to the structure of polynomials satisfying the required conditions given in Proposition 4.

Proposition 5. *Let* $I : [0, 1]^2 \to [0, 1]$ *be a fuzzy polynomial implication of degree* n. *Then there exists a polynomial* $q(x, y)$ *of degree* $n - 2$ *satisfying:*

- $q(1, 0) = -1$,
- $-q(x, y) + (1 - y)\frac{\partial q}{\partial y}(x, y) \ge 0$ *for all* $(x, y) \in [0, 1]^2$,
- $q(x, y) + x\frac{\partial q}{\partial x}(x, y) \le 0$ *for all* $(x, y) \in [0, 1]^2$,

such that $I(x, y) = x \cdot (1 - y) \cdot q(x, y) + 1$ *for all* $x, y \in [0, 1]$.

Proof. Let I be a fuzzy polynomial implication of degree n. Define $r(x, y) = I(x, y) - 1$ for all $x, y \in [0, 1]$. This polynomial satisfies

$$r(x, 1) = r(0, y) = 0, \quad \text{for all } x, y \in [0, 1],$$

since I satisfies $I(x, 1) = I(0, y) = 1$ for all $x, y \in [0, 1]$. Thus, r can be written as $r(x, y) = x \cdot (1 - y) \cdot q(x, y)$ where q is a bivariate polynomial of degree $n - 2$ with $q(1, 0) = r(1, 0) = I(1, 0) - 1 = -1$. Thus, $I(x, y) = r(x, y) + 1 = x \cdot (1 - y) \cdot q(x, y) + 1$. Now, since I is a fuzzy implication function, it satisfies the monotonocities and therefore, we obtain:

- $\frac{\partial I}{\partial y}(x, y) \ge 0 \Rightarrow -q(x, y) + (1 - y)\frac{\partial q}{\partial y}(x, y) \ge 0$ for all $(x, y) \in [0, 1]^2$.
- $\frac{\partial I}{\partial x}(x, y) \le 0 \Rightarrow q(x, y) + x\frac{\partial q}{\partial x}(x, y) \le 0$ for all $(x, y) \in [0, 1]^2$,

and the result follows. □

Consequently, the study of the coefficients of a fuzzy polynomial implication of degree n can be done by analysing the coefficients of the associated bivariate polynomial of degree $n - 2$ greatly reducing the complexity for low degrees.

4. On the characterisations of fuzzy polynomial implications of degree ≤ 4

This section is devoted to find all fuzzy polynomial implications of degree less or equal to 4. For such degrees, it is feasible to characterise the coefficients of the polynomial in order to ensure the fulfilment of the monotonicities required to a fuzzy implication function. Although the results involving smaller degrees can be derived as corollaries from the results involving greater degrees, for the sake of clarity, the study will be carried out step by step in an increasing order with respect to the degree of the polynomial. This procedure will clearly show how there is a high increase in complexity as the degree of the polynomial gets higher.

4.1. *Degrees zero and one*

As a first consequence of Proposition 5, the following result is straightforward.

Proposition 6. *There are no fuzzy polynomial implications of degree less or equal to 1.*

Proof. Given a fuzzy polynomial implication of degree n, Proposition 5 assures the existence of a polynomial $q(x, y)$ of degree $n - 2$ with $q(1, 0) = -1$ such that $I(x, y) = x \cdot (1 - y) \cdot q(x, y) + 1$. Thus, it must be $n \geq 2$. □

Consequently, there are no fuzzy polynomial implications with the following expressions:

$$\begin{aligned} I(x, y) &= a_{00}, && \text{with } a_{00} \in \mathbb{R}, \\ I(x, y) &= a_{00} + a_{10}x + a_{01}y, && \text{with } a_{10} \neq 0 \text{ or } a_{01} \neq 0. \end{aligned}$$

Again, note that there do exist fuzzy implication functions defined by a piecewise function whose components are polynomials of degree less or equal to one as $I_{\mathbf{GD}}$ and $I_{\mathbf{LK}}$. However, they are not fuzzy polynomial implications.

Example 1 ensures the existence of fuzzy polynomial implications of any degree $n \geq 2$. Therefore, let us study in detail those of degrees 2, 3 and 4.

4.2. *Degree two*

The following result proves that there is only one fuzzy polynomial implication of degree 2.

Proposition 7. *The Reichenbach implication $I_{\mathbf{RC}}$ is the unique fuzzy polynomial implication of degree* 2.

Proof. Consider $I : [0, 1]^2 \to [0, 1]$ a fuzzy polynomial implication of degree 2. Then, using Proposition 5, I is given by

$$I(x, y) = x \cdot (1 - y) \cdot (-1) + 1 = 1 - x + xy = I_{\mathbf{RC}}(x, y), \text{ for all } x, y \in [0, 1].$$

It is well known that $I_{\mathbf{RC}}$ satisfies Definition 2, see [3]. □

The Reichenbach implication is a well-known fuzzy implication function which belongs to the family of (S,N)-implications (it can be obtained as $I_{S_{\mathbf{P}}, N_{\mathbf{C}}}$) and also to the family of Yager's f-generated implications (taking $f(x) = 1 - x$ for all $x \in [0, 1]$). It satisfies **(LI)** with $T_{\mathbf{P}}$, **(EP)** and **(NP)** (see [3]). Moreover, note that it is obtained in Example 1 as $I_{S_{\mathbf{P}}, N_1}$.

4.3. *Degree three*

Let us study the fuzzy polynomial implications of degree 3. Next result shows that an infinity of such fuzzy implication functions exist.

Theorem 2. *Let $I : [0, 1]^2 \to [0, 1]$ be a binary operator. Then I is a fuzzy polynomial implication of degree 3 if, and only if, I is given by*

$$I_{\alpha,\beta}(x, y) = 1 + \alpha x + (-1 - \alpha)x^2 + (\beta - \alpha)xy + (1 + \alpha)x^2y - \beta xy^2 \tag{1}$$

with $\alpha, \beta \in \mathbb{R}$ such that $(\alpha, \beta) \neq (-1, 0)$ and one of these cases hold:

- *(i)* $-2 \leq \alpha \leq -1$ *and* $-1 \leq \beta \leq 2 + \alpha$.
- *(ii)* $-1 < \alpha \leq 0$ *and* $\alpha \leq \beta \leq -\alpha$.

Proof. First, consider I a fuzzy polynomial implication of degree 3. By Proposition 5, I is given by

$$I(x,y) = x \cdot (1-y) \cdot (\alpha + (-1-\alpha)x + \beta y) + 1$$

for all $x, y \in [0,1]$ with $(\alpha, \beta) \neq (-1, 0)$. Since I is a fuzzy implication function, it satisfies that:

- $\frac{\partial I(x,y)}{\partial x} \leq 0$ for all $x, y \in [0,1]$.
- $\frac{\partial I(x,y)}{\partial y} \geq 0$ for all $x, y \in [0,1]$.

The first inequality is equivalent to:

$$\alpha - 2 \cdot (1+\alpha)x + \beta y \leq 0$$

for all $x, y \in [0,1]$. Specifically, taking concrete values of x and y, we obtain the following inequalities:

- If $x = 0$ and $y = 0$, then $\alpha \leq 0$.
- If $x = 0$ and $y = 1$, then $\beta \leq -\alpha$.
- If $x = 1$ and $y = 0$, then $\alpha \geq -2$.
- If $x = 1$ and $y = 1$, then $\beta \leq 2 + \alpha$.

Similarly, if we consider the second inequality provided by the partial derivative, it is equivalent to

$$-\alpha + \beta + (1+\alpha)x - 2\beta y \geq 0$$

for all $x, y \in [0,1]$. Analogously, taking specific values of x and y, we get the following conditions:

- If $x = 0$ and $y = 0$, then $\beta \geq \alpha$.
- If $x = 0$ and $y = 1$, then $\beta \leq -\alpha$.
- If $x = 1$ and $y = 0$, then $\beta \geq -1$.
- If $x = 1$ and $y = 1$, then $\beta \leq 1$.

By joining all these conditions, we get that α and β are such that $(\alpha, \beta) \neq (-1, 0)$ and one of these cases hold:

(i) $-2 \leq \alpha \leq -1$ and $-1 \leq \beta \leq 2 + \alpha$.
(ii) $-1 < \alpha \leq 0$ and $\alpha \leq \beta \leq -\alpha$.

Conversely, let R' be the region of $\mathbb{R}^2$ given by the mentioned values (α, β). Let $R = R' \cup \{(-1, 0)\}$. Due to the convexity of R, each point $(\alpha, \beta) \in R$ is a linear convex combination of the vertices of R. Those polynomials $I_{\alpha,\beta}$ corresponding to the point $(\alpha, \beta) \in R$ can be expressed as the same linear convex combination of polynomials $I_{0,0}$, $I_{-1,1}$, $I_{-1,-1}$, $I_{-2,0}$ and $I_{-2,-1}$ obtained by means of the vertices of R. These polynomials are given by the following expressions:

- $I_{0,0}(x,y) = 1 - x^2 + x^2 y$.
- $I_{-1,1}(x,y) = 1 - x + 2xy - xy^2$.
- $I_{-1,-1}(x,y) = 1 - x + xy^2$.
- $I_{-2,0}(x,y) = 1 - 2x + x^2 + 2xy - x^2 y$.
- $I_{-2,-1}(x,y) = 1 - 2x + x^2 + xy - x^2 y + xy^2$.

In all these cases it is straightforward to check that all of them are fuzzy polynomial implications. As a linear convex combination of fuzzy implication functions is also a fuzzy implication function [3], $I_{\alpha,\beta}$ is a fuzzy polynomial implication for all $(\alpha, \beta) \in R$. Note, however, that the point $(-1, 0)$ must be removed because it generates $\boldsymbol{I_{\mathbf{RC}}}$, the unique fuzzy polynomial implication of degree 2. □

The region of the values of the parameters α and β in Expression (1) given in the previous theorem can be viewed in Fig. 1.

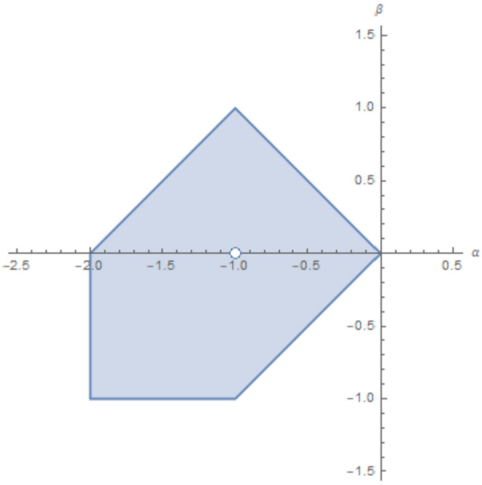


Fig. 1. Region of the values of the parameters α and β given in Theorem 2. Point $(-1,0)$ is not included in the region.

At this stage, let us study some additional properties of these fuzzy implication functions in order to analyse if they belong to any known family of these connectives.

Proposition 8. *Let $I_{\alpha,\beta}$ be a fuzzy polynomial implication of degree 3 given by Expression (1). Then the following statements are equivalent:*

- *(i)* $I_{\alpha,\beta}$ *satisfies* **(EP)**.
- *(ii)* $I_{\alpha,\beta}$ *satisfies* **(NP)**.
- *(iii)* $\beta = 0$.

In this case, the fuzzy polynomial implications are given by

$$I_{\alpha,0}(x,y) = 1 + \alpha x + (-1-\alpha)x^2 - \alpha xy + (1+\alpha)x^2 y \tag{2}$$

for all $x, y \in [0,1]$ *where* $\alpha \in [-2,0] \setminus \{-1\}$.

Proof. Let us start with (i)⇒(ii). If $I_{\alpha,\beta}$ satisfies **(EP)**, since $N_{I_{\alpha,\beta}}$ is continuous, according to Corollary 1.5.18 from [3], $I_{\alpha,\beta}$ satisfies **(NP)**.

For (ii)⇒(iii), if $I_{\alpha,\beta}$ satisfies **(NP)**, then from Expression (1), for all $y \in [0,1]$, we have that

$$y = I_{\alpha,\beta}(1,y) = (1+\beta)y - \beta y^2.$$

Therefore, it must hold that $\beta = 0$.

Finally, let us prove (iii)⇒(i). It is a matter of straightforward computation that

$$I_{\alpha,0}(x,y) = 1 + \alpha x + (-1-\alpha)x^2 - \alpha xy + (1+\alpha)x^2 y$$

for all $x, y \in [0,1]$ where $\alpha \in [-2,0] \setminus \{-1\}$ satisfies **(EP)**. □

Since the most usual fuzzy implications such as (S,N), R, QL or Yager's f and g-generated implications satisfy **(NP)**, the previous result proves that there exist fuzzy polynomial implications of degree 3 which do not belong to any of the aforementioned families. For example, the fuzzy polynomial implications $I_{-1,1}$, $I_{-1,-1}$ and $I_{-2,-1}$ given in the proof of Theorem 2 do not satisfy **(NP)**.

On the other hand, using Proposition 3, those fuzzy polynomial implications of degree 3 satisfying **(EP)** are (S,N)-implications obtained from the unique polynomial t-conorm S_P.

Theorem 3. *Let $I : [0,1]^2 \to [0,1]$ be a binary operator, S a t-conorm and N a fuzzy negation. Then the following statements are equivalent:*

(i) I is a fuzzy polynomial implication of degree 3 and an (S, N)-implication obtained from S and N.
(ii) $S = S_{\mathbf{P}}$ and $N(x) = 1 + \alpha x + (-1-\alpha)x^2$ with $\alpha \in [-2, 0] \setminus \{-1\}$.

Proof. The proof is straightforward using Propositions 3 and 8. □

Finally, and using the characterization of Yager's f-generated implications with $f(0) < +\infty$ in [12, Theorem 6], the next result determines which fuzzy polynomial implications of degree 3 belong to this Yager's family.

Theorem 4. *Let $I : [0,1]^2 \to [0,1]$ be a binary operator. Then the following assertions are equivalent:*

(i) I is a fuzzy polynomial implication of degree 3 and a Yager's f-generated implication with $f(0) < +\infty$.
(ii) I is a fuzzy polynomial implication of degree 3 satisfying **(LI)** *with respect to the product t-norm $T_{\mathbf{P}}$.*
(iii) I is given by $I(x, y) = 1 - x^2 + x^2 y$, which is the f-generated implication with $f(x) = \sqrt{1-x}$ for all $x, y \in [0,1]$.

Proof. (i)⇔(ii) is straightforward taking into account that N_I is a continuous fuzzy negation and Theorem 6 in [12].

(iii)⇒(ii) can be also easily proved. Indeed, it is clear that $I(x, y) = 1 - x^2 + x^2 y$ is a fuzzy polynomial implication of degree 3 since it is a bivariate polynomial of degree 3 being also the f-generated implication with $f(x) = \sqrt{1-x}$ (and in particular, a fuzzy implication function). Moreover, the fulfilment of **(LI)** with respect to $T_{\mathbf{P}}$ is ensured by Theorem 6 in [12].

Finally, let us prove (ii)⇒(iii). If I is a fuzzy polynomial implication of degree 3 satisfying **(LI)** with respect to the product t-norm $T_{\mathbf{P}}$, it also satisfies **(EP)** and it is given by Equation (2). Now, if it satisfies **(LI)** with respect to $T_{\mathbf{P}}$, it holds that

$$\begin{aligned}1 + \alpha xy + (-1-\alpha)x^2y^2 = I(xy, 0) &= I(x, I(y, 0)) \\ &= 1 - \alpha^2 xy + \alpha(1+\alpha)x^2 y + \alpha(1+\alpha)xy^2 + (-1-2\alpha-\alpha^2)x^2y^2\end{aligned}$$

for all $x, y \in [0,1]$. Thus, $\alpha = 0$ or $\alpha = -1$. Since $\alpha = -1$ retrieves $I_{\mathbf{RC}}$, the unique fuzzy polynomial implication of degree 2, the unique solution of degree 3 is given by $\alpha = 0$, which retrieves $I(x, y) = 1 - x^2 + x^2 y$. By Theorem 6 in [12], this fuzzy implication function is the f-generated implication given by $f(x) = N_I^{-1}(x) = \sqrt{1-x}$. □

The fuzzy implication function obtained in the previous result belongs to the family considered in Example 1 taking I_{S_P, N_2}.

4.4. Degree four

The case of fuzzy polynomial implications of degree 4 is much more complex. Indeed, first of all, several results involving polynomials defined in [0, 1] will be proved. These results will be used later in order to manage the monotonicities of the fuzzy polynomial implications.

Lemma 1. *Let $p(x) = a_0 + a_1 x + a_2 x^2$ be a polynomial of one variable of degree ≤ 2. Then $p(x) \leq 0$ for all $x \in [0,1]$ if and only if the following conditions hold simultaneously:*

(1) $a_0 \leq 0$,
(2) $a_0 + a_1 + a_2 \leq 0$,
(3) *If $a_2 < 0$ and $0 \leq a_1 \leq -2a_2$, then $a_1^2 - 4a_0 a_2 \leq 0$.*

Proof. Let us suppose first that $p(x) \leq 0$ for all $x \in [0,1]$. Then $p(0) = a_0 \leq 0$ and $p(1) = a_0 + a_1 + a_2 \leq 0$ and Conditions (1) and (2) are obtained. Now, suppose that $a_2 < 0$ and $0 \leq a_1 \leq -2a_2$, in this case the polynomial p has a maximum at $-\dfrac{a_1}{2a_2}$ within [0, 1] since $0 \le a_1 \le -2a_2$. Moreover, the image of this maximum is also non-positive and therefore,

$$p\left(-\frac{a_1}{2a_2}\right) = a_0 - \frac{a_1^2}{2a_2} + \frac{a_1^2}{4a_2} \le 0 \Rightarrow a_1^2 - 4a_0a_2 \le 0$$

and Condition (3) is obtained.

Reciprocally, suppose that Conditions (1)-(3) are satisfied. By Conditions (1) and (2), $p(0) = a_0 \le 0$ and $p(1) = a_0 + a_1 + a_2 \le 0$. Now two cases arise. If $a_2 = 0$, then p is a bivariate polynomial of degree ≤ 1 and it follows that $p(x) \le 0$ for all $x \in [0, 1]$. Otherwise, if $a_2 \neq 0$, it is clear that if p has a minimum or the maximum $-\dfrac{a_1}{2a_2}$ does not belong to [0, 1], then $p(x) \le 0$ for all $x \in [0, 1]$. Therefore, the only problematic case would be when the maximum $-\dfrac{a_1}{2a_2}$ belongs to [0, 1] which is equivalent to $0 \le a_1 \le -2a_2$. However, Condition (3) ensures that the image of this maximum is negative and consequently, again $p(x) \le 0$ for all $x \in [0, 1]$. □

Corollary 1. *Let $p(x) = a_0 + a_1x + a_2x^2$ be a polynomial of one variable of degree ≤ 2. Then $p(x) \ge 0$ for all $x \in [0, 1]$ if and only if the following conditions hold simultaneously:*

(1) $a_0 \ge 0$,
(2) $a_0 + a_1 + a_2 \ge 0$,
(3) *If $a_2 > 0$ and $-2a_2 \le a_1 \le 0$, then $a_1^2 - 4a_0a_2 \le 0$.*

Proof. Just consider $-p(x)$ and apply Lemma 1. □

Now, let us follow with similar results but now for polynomials of two variables.

Lemma 2. *Let $p(x, y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$ be a bivariate polynomial of degree ≤ 2 of two variables. Then $p(x, y) \le 0$ for all $(x, y) \in [0, 1]^2$ if and only if the following conditions hold simultaneously:*

(1) $a_{00} \le 0$,
(2) $a_{00} + a_{10} + a_{20} \le 0$,
(3) $a_{00} + a_{01} + a_{02} \le 0$,
(4) $a_{00} + a_{10} + a_{01} + a_{11} + a_{20} + a_{02} \le 0$,
(5) *If $a_{20} < 0$ and $0 \le a_{10} \le -2a_{20}$, then $a_{10}^2 - 4a_{00}a_{20} \le 0$,*
(6) *If $a_{20} < 0$ and $0 \le a_{10} + a_{11} \le -2a_{20}$, then $(a_{10} + a_{11})^2 - 4a_{20}(a_{00} + a_{01} + a_{02}) \le 0$,*
(7) *If $a_{02} < 0$ and $0 \le a_{01} \le -2a_{02}$, then $a_{01}^2 - 4a_{00}a_{02} \le 0$,*
(8) *If $a_{02} < 0$ and $0 \le a_{01} + a_{11} \le -2a_{02}$, then $(a_{01} + a_{11})^2 - 4a_{02}(a_{00} + a_{10} + a_{20}) \le 0$.*
(9) *If $a_{20} < 0$, $a_{02} < 0$, $4a_{20}a_{02} - a_{11}^2 > 0$, $0 \le a_{01}a_{11} - 2a_{10}a_{02} \le 4a_{20}a_{02} - a_{11}^2$ and $0 \le a_{10}a_{11} - 2a_{01}a_{20} \le 4a_{02}a_{20} - a_{11}^2$, then $a_{00} - \frac{a_{01}^2a_{20} + a_{10}^2a_{02} - a_{10}a_{01}a_{11}}{4a_{20}a_{02} - a_{11}^2} \le 0$.*
(10) *If $a_{20} < 0$, $a_{02} < 0$, $a_{11}^2 = 4a_{20}a_{02}$, $a_{01}a_{11} - 2a_{10}a_{02} = 0$ and one or more of the following conditions hold:*
  (a) $-1 \le \frac{a_{10}}{a_{11}} \le 0$,
  (b) $0 \le a_{10} \le -2a_{20}$,
  (c) $-1 \le \frac{a_{10} + 2a_{20}}{a_{11}} \le 0$,
  (d) $0 \le a_{10} + a_{11} \le -2a_{20}$,
  then $a_{00} - \frac{a_{10}^2}{4a_{20}} \le 0$.

Proof. The condition $p(x, y) \le 0$ for all $(x, y) \in [0, 1]^2$ implies first of all that $p(x, y) \le 0$ in the boundary of $[0, 1]^2$, that is:

$$p(x, 0) \le 0, \ \ p(x, 1) \le 0, \ \ p(0, y) \le 0, \ \ p(1, y) \le 0, \quad \text{for all } (x, y) \in [0, 1]^2.$$

The polynomial $p(x,0)$ is given by $p(x,0) = a_{20}x^2 + a_{10}x + a_{00}$. Using Lemma 1, $p(x,0) \leq 0$ is equivalent to the following conditions:

(1) $a_{00} \leq 0$,
(2) $a_{20} + a_{10} + a_{00} \leq 0$,
(3) If $a_{20} < 0$ and $0 \leq a_{10} \leq -2a_{20}$, then $a_{10}^2 - 4a_{20}a_{00} \leq 0$.

The polynomial $p(x,1)$ is given by $p(x,1) = a_{20}x^2 + (a_{11} + a_{10})x + a_{02} + a_{01} + a_{00}$. Using again Lemma 1, $p(x,1) \leq 0$ is equivalent to the following conditions:

(1) $a_{02} + a_{01} + a_{00} \leq 0$,
(2) $a_{20} + a_{02} + a_{11} + a_{10} + a_{01} + a_{00} \leq 0$,
(3) If $a_{20} < 0$ and $0 \leq a_{11} + a_{10} \leq -2a_{20}$, then $(a_{11} + a_{10})^2 - 4a_{20}(a_{02} + a_{01} + a_{00}) \leq 0$.

The polynomial $p(0,y)$ is given by $p(0,y) = a_{02}y^2 + a_{01}y + a_{00}$. Using Lemma 1, $p(0,y) \leq 0$ is equivalent to the following conditions:

(1) $a_{00} \leq 0$,
(2) $a_{02} + a_{01} + a_{00} \leq 0$,
(3) If $a_{02} < 0$ and $0 \leq a_{01} \leq -2a_{02}$, then $a_{01}^2 - 4a_{02}a_{00} \leq 0$.

The polynomial $p(1,y)$ is given by $p(1,y) = a_{02}y^2 + (a_{11} + a_{01})y + a_{20} + a_{10} + a_{00}$. Using Lemma 1, $p(1,y) \leq 0$ is equivalent to the following conditions:

(1) $a_{20} + a_{10} + a_{00} \leq 0$,
(2) $a_{20} + a_{02} + a_{11} + a_{10} + a_{01} + a_{00} \leq 0$,
(3) If $a_{02} < 0$ and $0 \leq a_{11} + a_{01} \leq -2a_{02}$, then $(a_{11} + a_{01})^2 - 4a_{02}(a_{20} + a_{10} + a_{00}) \leq 0$.

Collecting the previous conditions and removing the duplicated ones, it holds that $p(x,y) \leq 0$, for all (x,y) in the boundary of $[0,1]^2$ if and only if the following conditions are true:

(1) $a_{00} \leq 0$,
(2) $a_{20} + a_{10} + a_{00} \leq 0$,
(3) $a_{02} + a_{01} + a_{00} \leq 0$,
(4) $a_{20} + a_{02} + a_{11} + a_{10} + a_{01} + a_{00} \leq 0$,
(5) If $a_{20} < 0$ and $0 \leq a_{10} \leq -2a_{20}$, then $a_{10}^2 - 4a_{20}a_{00} \leq 0$,
(6) If $a_{20} < 0$ and $0 \leq a_{11} + a_{10} \leq -2a_{20}$, then $(a_{11} + a_{10})^2 - 4a_{20}(a_{02} + a_{01} + a_{00}) \leq 0$,
(7) If $a_{02} < 0$ and $0 \leq a_{01} \leq -2a_{02}$, then $a_{01}^2 - 4a_{02}a_{00} \leq 0$,
(8) If $a_{02} < 0$ and $0 \leq a_{11} + a_{01} \leq -2a_{02}$, then $(a_{11} + a_{01})^2 - 4a_{02}(a_{20} + a_{10} + a_{00}) \leq 0$.

Once the boundary has been analysed, it remains to impose that in the case that the polynomial has a maximum and it belongs to $[0,1]^2$, its value must be negative. The eigenvalues of the Hessian matrix of the function $p(x,y)$ are:

$$\lambda_{\pm} = a_{20} + a_{02} \pm \sqrt{(a_{20} - a_{02})^2 + a_{11}^2}.$$

The polynomial $p(x,y)$ will have a maximum if and only if the previous eigenvalues are negative, that is, if and only if $a_{20} < 0,\ a_{02} < 0$ and $4a_{20}a_{02} \geq a_{11}^2$. We note that a_{20} and a_{02} cannot be zero because in that case, one of the previous eigenvalues would be positive. At this point, the extreme values of the polynomial $p(x,y)$ are computed through the following system of equations:

$$\left.\begin{aligned} 2a_{20}x + a_{11}y &= -a_{10}, \\ a_{11}x + 2a_{02}y &= -a_{01}. \end{aligned}\right\}$$

First, we suppose that $4a_{20}a_{02} > a_{11}^2$. In this case, the solution of the previous system is:

$$\tilde{x} = \frac{a_{11}a_{01} - 2a_{02}a_{10}}{4a_{20}a_{02} - a_{11}^2}, \quad \tilde{y} = \frac{2a_{20}a_{01} - a_{11}a_{10}}{a_{11}^2 - 4a_{20}a_{02}}.$$

The extreme value $(\tilde{x}, \tilde{y})$ belongs to $[0, 1]^2$ (taking into account that $4a_{20}a_{02} - a_{11}^2 > 0$) if and only if

$$\begin{aligned} 0 \leq & a_{11}a_{01} - 2a_{02}a_{10} \leq 4a_{20}a_{02} - a_{11}^2, \\ 0 \leq & a_{11}a_{10} - 2a_{20}a_{01} \leq 4a_{20}a_{02} - a_{11}^2, \end{aligned}$$

and $p(\tilde{x}, \tilde{y}) \leq 0$ if and only if

$$a_{00} - \frac{a_{20}a_{01}^2 + a_{02}a_{10}^2 - a_{11}a_{10}a_{01}}{4a_{20}a_{02} - a_{11}^2} \leq 0.$$

Finally, we suppose that $4a_{20}a_{02} - a_{11}^2 = 0$. In this case, if $a_{11}a_{01} - 2a_{02}a_{10} = 0$ or $-2a_{20}a_{01} + a_{10}a_{11} = 0$, there is an infinity of maxima, all located on the line $2a_{20}x + a_{11}y = -a_{10}$. This line intersects the domain $[0, 1]^2$ if and only if one or more of the following conditions hold:

(1) $-1 \leq \frac{a_{10}}{a_{11}} \leq 0,$
(2) $-1 \leq \frac{a_{10}}{2a_{20}} \leq 0, \Rightarrow 0 \leq a_{10} \leq -2a_{20},$
(3) $-1 \leq \frac{a_{10}+2a_{20}}{a_{11}} \leq 0,$
(4) $-1 \leq \frac{a_{11}+a_{10}}{2a_{20}} \leq 0, \Rightarrow 0 \leq a_{11} + a_{10} \leq -2a_{20}.$

In this case, the value of $p(x, y)$ for all (x, y) belonging to the previous line is: $a_{00} - \frac{a_{10}^2}{4a_{20}}$. So, it is necessary that $a_{00} - \frac{a_{10}^2}{4a_{20}} \leq 0$.

To sum up, $p(x, y)$ has a maximum in the domain $[0, 1]^2$ whose value is negative if and only if one of the following conditions holds:

(a) If $a_{20} < 0,\ a_{02} < 0,\ 4a_{20}a_{02} - a_{11}^2 > 0$, $0 \leq a_{11}a_{01} - 2a_{02}a_{10} \leq 4a_{20}a_{02} - a_{11}^2$ and $0 \leq a_{11}a_{10} - 2a_{20}a_{01} \leq 4a_{20}a_{02} - a_{11}^2$, then $a_{00} - \frac{a_{20}a_{01}^2 + a_{02}a_{10}^2 - a_{11}a_{10}a_{01}}{4a_{20}a_{02} - a_{11}^2} \leq 0$.
(b) If $a_{20} < 0,\ a_{02} < 0,\ a_{11}^2 = 4a_{20}a_{02}$, $a_{11}a_{01} - 2a_{02}a_{10} = 0$ and one or more of the conditions (1), (2), (3) and (4) hold, then $a_{00} - \frac{a_{10}^2}{4a_{20}} \leq 0$. □

At this point, the complete characterisation of all fuzzy polynomial implications of degree 4 is presented. This result provides the general expression of these operators which depends on five parameters satisfying up to 19 conditions.

Theorem 5. *Let $I : [0, 1]^2 \to [0, 1]$ be a binary operator. Then I is a fuzzy polynomial implication of degree 4 if and only if I is given by*

$$I(x, y) = x(1 - y)(\alpha(x - 1)^2 + \beta(x - 1) + \gamma y^2 + \delta xy + \epsilon y - 1) + 1 \tag{3}$$

with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ such that $(\alpha, \gamma, \delta) \neq (0, 0, 0)$ and the following conditions are satisfied:

(1). $\alpha - \beta - \epsilon \leq 1,$
(2). $\delta + \epsilon + 1 \geq 0,$
(3). $\alpha - \beta + \gamma + \epsilon \leq 1,$
(4). $\gamma + \delta + \epsilon \leq 1,$
(5). *If $\alpha < 0$ and $-2\alpha \leq \beta - \delta \leq -4\alpha$, then $(\beta - \delta)^2 + 4\alpha(\delta + \epsilon + 1) \leq 0$,*
(6). *If $\alpha < 0$ and $2\alpha \leq \beta + \delta \leq 0$, then $(\beta + \delta)^2 - 4a(\gamma + \delta + \epsilon - 1) \leq 0$,*
(7). *If $\gamma < 0$ and $\gamma \leq \epsilon \leq -2\gamma$, then $\gamma^2 + \epsilon^2 + \gamma(-3\alpha + 3\beta + \epsilon + 3) \leq 0$,*

(8). *If* $\gamma < 0$ *and* $\gamma \leq \delta + \epsilon \leq -2\gamma$, *then* $\gamma^2 + (\delta+\epsilon)^2 + \gamma(\delta+\epsilon+3) \leq 0$.

(9). *If* $\alpha < 0,\ \gamma < 0,\ 3\alpha\gamma - \delta^2 > 0,\ 2\delta^2 \leq 3\beta\gamma - \delta(\gamma + 2\epsilon) \leq 6\alpha\gamma$ *and* $\delta^2 - 4\alpha\gamma \leq -\beta\delta + 2\alpha(\delta+\epsilon) \leq 2\alpha\gamma - \delta^2$ *then* $-4\alpha\left(\gamma^2 + \gamma(\delta+\epsilon+3) + (\delta+\epsilon)^2\right) - 3\beta^2\gamma + 2\beta\delta(\gamma + 2(\delta+\epsilon)) + (\gamma+4)\delta^2 \leq 0$.

(10). *If* $\alpha < 0,\ \gamma < 0,\ \delta^2 = 3\alpha\gamma,\ 6\alpha(\delta+\epsilon) + \delta(\delta - 3\beta) = 0$, *and one or more of the following conditions hold:*
  (a) $-1 \leq \frac{\beta - 2\alpha}{\delta} \leq 1$,
  (b) $2\alpha \leq \beta - \delta \leq 0$,
  (c) $-1 \leq \frac{\beta}{\delta} \leq 1$,
  (d) $2\alpha \leq \beta + \delta \leq 0$,
  then $4\alpha(\delta+\epsilon+1) + (\beta-\delta)^2 \leq 0$.

(11). $\alpha - \beta \leq 1$,

(12). $\beta \leq 1$,

(13). $\beta + \gamma + 2\delta + \epsilon \leq 1$,

(14). *If* $\alpha < 0$ *and* $2\alpha \leq \beta \leq -\alpha$, *then* $\alpha^2 + \beta^2 - \alpha\beta + 3\alpha \leq 0$,

(15). *If* $\alpha < 0$ *and* $2\alpha \leq \beta + \delta \leq -\alpha$, *then* $\alpha^2 - \alpha(\beta + 3\gamma + 4\delta + 3\epsilon - 3) + (\beta+\delta)^2 \leq 0$,

(16). *If* $\gamma < 0$ *and* $0 \leq \epsilon \leq -2\gamma$, *then* $\epsilon^2 + 4\gamma(-\alpha + \beta + 1) \leq 0$,

(17). *If* $\gamma < 0$ *and* $0 \leq 2\delta + \epsilon \leq -2\gamma$, *then* $(2\delta+\epsilon)^2 - 4(\beta - 1)\gamma \leq 0$.

(18). *If* $\alpha < 0,\ \gamma < 0,\ 3\alpha\gamma - \delta^2 > 0,\ -4\alpha\gamma \leq -2\beta\gamma + \delta\epsilon \leq 2\alpha\gamma - 2\delta^2$ *and* $0 \leq -4\alpha\delta + 2\beta\delta - 3\alpha\epsilon \leq 6\alpha\gamma - 2\delta^2$ *then* $-4\alpha^2\gamma + 4\alpha(\beta-3)\gamma - \alpha(2\delta+\epsilon)(2\delta+3\epsilon) - 4\beta^2\gamma + 4\beta\delta(\delta+\epsilon) + 4\delta^2 \leq 0$.

(19). *If* $\alpha < 0,\ \gamma < 0,\ \delta^2 = 3\alpha\gamma,\ 4\alpha\delta + 3\alpha\epsilon - 2\beta\delta = 0$, *and one or more of the following conditions hold:*

  (a) $-1 \leq \frac{\beta - 2\alpha}{\delta} \leq 0$,
  (b) $2\alpha \leq \beta \leq -\alpha$,
  (c) $-1 \leq \frac{\alpha+\beta}{\delta} \leq 0$,
  (d) $2\alpha \leq \beta + \delta \leq -\alpha$,

  then $\alpha^2 + \beta^2 - \alpha\beta + 3\alpha \leq 0$.

Proof. Let us suppose that I is a fuzzy polynomial implication of degree 4. Then according to Proposition 5, there exists a polynomial $p(x,y)$ of degree 2 satisfying the following conditions:

- $p(1,0) = -1$,
- $-p(x,y) + (1-y)\frac{\partial p}{\partial y}(x,y) \geq 0$ for all $(x,y) \in [0,1]^2$,
- $p(x,y) + x\frac{\partial p}{\partial x}(x,y) \leq 0$ for all $(x,y) \in [0,1]^2$,

and such that

$$I(x,y) = x(1-y)p(x,y) + 1.$$

Using the basis $\{(x-1)^2, y^2, xy, x-1, y, 1\}$ to express the polynomial p, we obtain that

$$I(x,y) = x(1-y)(\alpha(x-1)^2 + \beta(x-1) + \gamma y^2 + \delta xy + \epsilon y - 1) + 1,$$

for some $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ such that $(\alpha, \gamma, \delta) \neq (0,0,0)$. Note that the constant term must be always -1 due to the fact that $p(1,0) = -1$.

Now, since I is a fuzzy implication function, it is decreasing in the first variable. Therefore, it holds that:

$$\begin{aligned}
\frac{\partial I}{\partial x}(x,y) &= (1-y)(\alpha(x-1)^2 + \beta(x-1) + \gamma y^2 + \delta xy + \epsilon y - 1) + x(1-y)(2\alpha(x-1) + \beta + \delta y) \\
&= (1-y)\cdot(3\alpha x^2 + \gamma y^2 + 2\delta xy + 2(\beta - 2\alpha)x + \epsilon y + \alpha - \beta - 1) \leq 0 \\
&\Rightarrow 3\alpha x^2 + \gamma y^2 + 2\delta xy + 2(\beta - 2\alpha)x + \epsilon y + \alpha - \beta - 1 \leq 0.
\end{aligned}$$

By Lemma 2 this last inequality is equivalent to the following conditions:

(1) $\alpha - \beta \leq 1$,
(2) $\beta \leq 1$,

(3) $\alpha - \beta + \gamma + \epsilon \leq 1$,
(4) $\beta + \gamma + 2\delta + \epsilon \leq 1$,
(5) If $\alpha < 0$ and $2\alpha \leq \beta \leq -\alpha$, then $\alpha^2 + \beta^2 - \alpha\beta + 3\alpha \leq 0$,
(6) If $\alpha < 0$ and $2\alpha \leq \beta + \delta \leq -\alpha$, then $\alpha^2 - \alpha(\beta + 3\gamma + 4\delta + 3\epsilon - 3) + (\beta + \delta)^2 \leq 0$,
(7) If $\gamma < 0$ and $0 \leq \epsilon \leq -2\gamma$, then $\epsilon^2 + 4\gamma(-\alpha + \beta + 1) \leq 0$,
(8) If $\gamma < 0$ and $0 \leq 2\delta + \epsilon \leq -2\gamma$, then $(2\delta + \epsilon)^2 - 4(\beta - 1)\gamma \leq 0$.
(9) If $\alpha < 0,\ \gamma < 0,\ 3\alpha\gamma - \delta^2 > 0,\ -4\alpha\gamma \leq -2\beta\gamma + \delta\epsilon \leq 2\alpha\gamma - 2\delta^2$ and $0 \leq -4\alpha\delta + 2\beta\delta - 3\alpha\epsilon \leq 6\alpha\gamma - 2\delta^2$ then $-4\alpha^2\gamma + 4\alpha(\beta - 3)\gamma - \alpha(2\delta + \epsilon)(2\delta + 3\epsilon) - 4\beta^2\gamma + 4\beta\delta(\delta + \epsilon) + 4\delta^2 \leq 0$.
(10) If $\alpha < 0,\ \gamma < 0,\ \delta^2 = 3\alpha\gamma,\ \delta\epsilon - 2\beta\gamma + 4\alpha\gamma = 0$ and one or more of the following conditions hold:
  (a) $-1 \leq \frac{\beta - 2\alpha}{\delta} \leq 0$,
  (b) $2\alpha \leq \beta \leq -\alpha$,
  (c) $-1 \leq \frac{\alpha + \beta}{\delta} \leq 0$,
  (d) $2\alpha \leq \beta + \delta \leq -\alpha$,
  then $\alpha^2 + \beta^2 - \alpha\beta + 3\alpha \leq 0$.

Now, since I is a fuzzy implication function, it is increasing in the second variable. Therefore, it holds that:

$$\begin{aligned}\frac{\partial I}{\partial y}(x, y) &= -x(\alpha(x-1)^2 + \beta(x-1) + \gamma y^2 + \delta xy + \epsilon y - 1) + x(1-y)(2\gamma y + \delta x + \epsilon) \geq 0\\ &\Rightarrow \alpha x^2 + 3\gamma y^2 + 2\delta xy + (-2\alpha + \beta - \delta)x + 2(\epsilon - \gamma) + \alpha - \beta - \epsilon - 1 \leq 0.\end{aligned}$$

Again, by Lemma 2, this last inequality is equivalent to the following conditions:

(11) $\alpha - \beta - \epsilon \leq 1$,
(12) $\delta + \epsilon + 1 \geq 0$,
(3) $\alpha - \beta + \gamma + \epsilon \leq 1$,
(13) $\gamma + \delta + \epsilon \leq 1$,
(14) If $\alpha < 0$ and $-2\alpha \leq \beta - \delta \leq -4\alpha$, then $(\beta - \delta)^2 + 4\alpha(\delta + \epsilon + 1) \leq 0$,
(15) If $\alpha < 0$ and $2\alpha \leq \beta + \delta \leq 0$, then $(\beta + \delta)^2 - 4\alpha(\gamma + \delta + \epsilon - 1) \leq 0$,
(16) If $\gamma < 0$ and $\gamma \leq \epsilon \leq -2\gamma$, then $\gamma^2 + \epsilon^2 + \gamma(-3\alpha + 3\beta + \epsilon + 3) \leq 0$,
(17) If $\gamma < 0$ and $\gamma \leq \delta + \epsilon \leq -2c$, then $\gamma^2 + (\delta + \epsilon)^2 + \gamma(\delta + \epsilon + 3) \leq 0$.
(18) If $\alpha < 0,\ \gamma < 0,\ 3\alpha\gamma - \delta^2 > 0,\ 2\delta^2 \leq 3\beta\gamma - \delta(\gamma + 2\epsilon) \leq 6\alpha\gamma$ and $\delta^2 - 4\alpha\gamma \leq -\beta\delta + 2\alpha(\delta + \epsilon) \leq 2\alpha\gamma - \delta^2$ then $-4\alpha\left(\gamma^2 + \gamma(\delta + \epsilon + 3) + (\delta + \epsilon)^2\right) - 3\beta^2\gamma + 2\beta\delta(\gamma + 2(\delta + \epsilon)) + (\gamma + 4)\delta^2 \leq 0$.
(19) If $\alpha < 0,\ \gamma < 0,\ \delta^2 = 3\alpha\gamma,\ \gamma\delta + 2\epsilon\delta + 6\alpha\gamma - 3\beta\gamma = 0$ and one or more of the following conditions hold:
  (a) $-1 \leq \frac{\beta - 2\alpha}{\delta} \leq 1$,
  (b) $2\alpha \leq \beta - \delta \leq 0$,
  (c) $-1 \leq \frac{\beta}{\delta} \leq 1$,
  (d) $2\alpha \leq \beta + \delta \leq 0$,
  then $4\alpha(\delta + \epsilon + 1) + (\beta - \delta)^2 \leq 0$.

This concludes the first part of the proof.

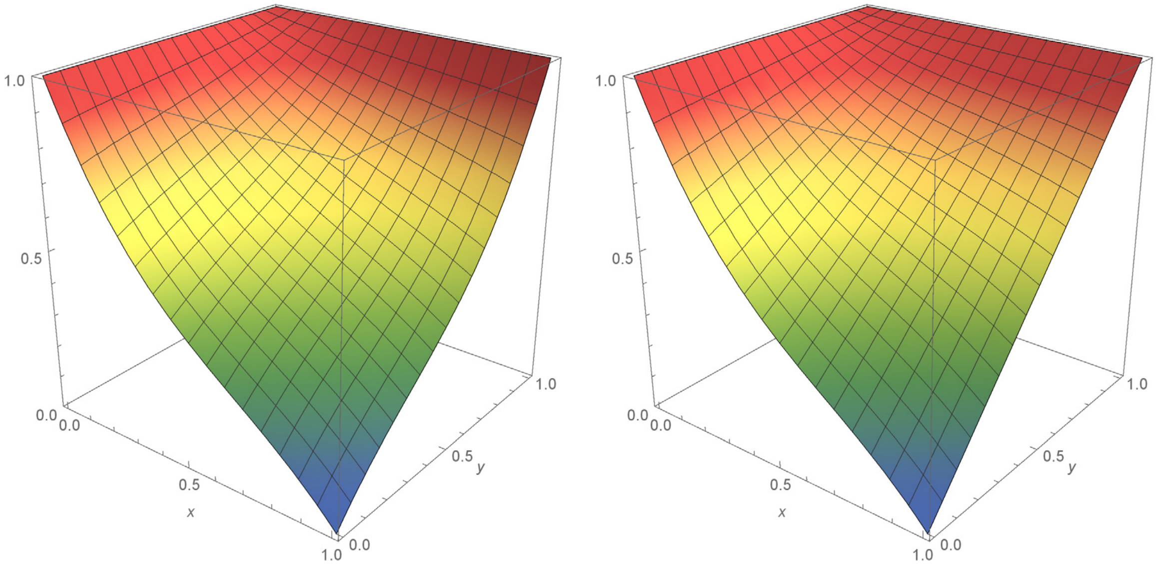
Reciprocally, let us consider that I is given by:

$$I(x, y) = x(1-y)(\alpha(x-1)^2 + \beta(x-1) + \gamma y^2 + \delta xy + \epsilon y - 1) + 1$$

with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ such that $(\alpha, \gamma, \delta) \neq (0, 0, 0)$ and the Conditions (1)-(19) are satisfied. We have to prove that I is a fuzzy polynomial implication of degree 4.

First, it is clear that I is a bivariate polynomial of degree 4. Now, since the coefficients fulfil Conditions (1)-(10), by using Lemma 2, we have that $\frac{\partial I}{\partial x} \leq 0$ for all $(x, y) \in [0, 1]^2$ and therefore, I is decreasing in the first variable. Analogously, since the coefficients satisfy Conditions (3) and (11)-(19), by using Lemma 2, we have that $\frac{\partial I}{\partial y} \geq 0$ for all $(x, y) \in [0, 1]^2$ and therefore, I is increasing in the second variable.

Finally, it is straightforward to check that $I(0, y) = 1$ for all $y \in [0, 1]$, $I(x, 1) = 1$ for all $x \in [0, 1]$ and $I(1, 0) = 0$. These last conditions, jointly with the monotonicities, imply that $0 \leq I(x, y) \leq 1$ for all $(x, y) \in [0, 1]^2$ and the result follows. □



(a) $(-1, 0, -1, 0, 0)$ (b) $(-1, 0, 0, 0, 0)$

Fig. 2. Two examples of fuzzy polynomial implications of degree 4 for different values of $(\alpha, \beta, \gamma, \delta, \epsilon)$.

Let us provide some examples of fuzzy polynomial implications of degree 4.

Example 2.

(a) Let us consider in Equation (3) the following parameter values: $\alpha = -1$, $\gamma = -1$ and $\beta = \delta = \epsilon = 0$. These values satisfy the conditions given by Theorem 5 and therefore, the operator

$$I(x, y) = x(1-y)(-(x-1)^2 - y^2 - 1) + 1$$

is a fuzzy polynomial implication of degree 4.

(b) Similarly, we can choose $\alpha = -1$ and $\beta = \gamma = \delta = \epsilon = 0$, which also satisfy all the conditions, retrieving the following fuzzy polynomial implication of degree 4:

$$I(x, y) = x(1-y)(-(x-1)^2 - 1) + 1.$$

Both fuzzy implication functions are depicted in Fig. 2. Note that the second implication satisfies **(NP)** while the first one does not.

Although Theorem 5 is an important result which solves the goal of this section, the large number of conditions makes difficult to find parameter values of $(\alpha, \beta, \gamma, \delta, \epsilon)$ satisfying all the conditions and consequently, providing examples of fuzzy polynomial implications is not an easy task. To overcome this problem, next result defines a convex polyhedron in $\mathbb{R}^5$ which defines a whole family of fuzzy polynomial implications of degree 4.

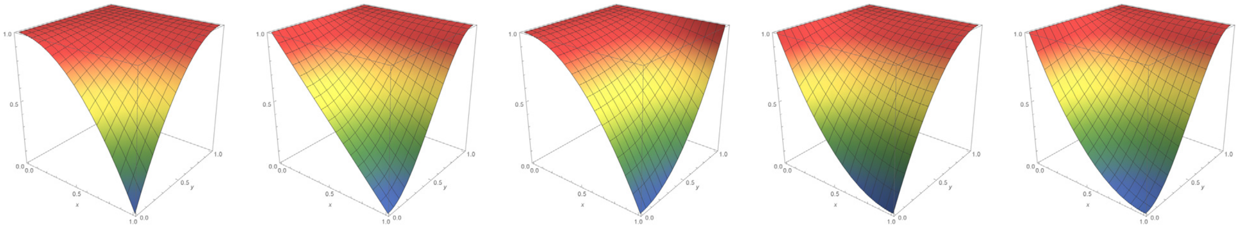
Theorem 6. *Let* $I : [0, 1]^2 \to [0, 1]$ *be a binary operator given by*

$$I(x, y) = x(1-y)(\alpha(x-1)^2 + \beta(x-1) + \gamma y^2 + \delta xy + \epsilon y - 1) + 1$$

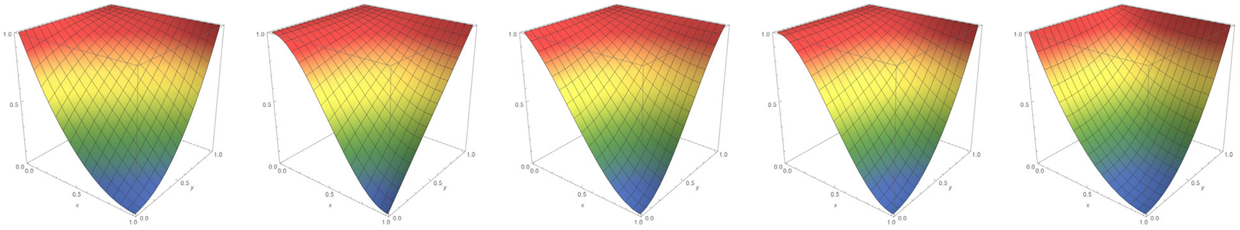
with $(\alpha, \beta, \gamma, \delta, \epsilon) \subset \mathbb{R}^5$ *such that*

$$(\alpha, \beta, \gamma, \delta, \epsilon) = \sum_{i=1}^{10} \lambda_i V_i, \quad \lambda_i \in [0, 1], \quad \sum_{i=1}^{10} \lambda_i = 1,$$

and



(a) $(0,-1,0,1,0)$ (b) $(0,0,2,0,-1)$ (c) $(0,-1,0,-1,0)$ (d) $(0,1,0,-1,2)$ (e) $(0,1,2,-1,0)$



(f) $(0,1,0,-3,2)$ (g) $(2,1,0,0,0)$ (h) $\left(\frac{4}{3},1,\frac{4}{3},-\frac{1}{3},-\frac{2}{3}\right)$ (i) $(2,1,0,-1,0)$ (j) $(0,1,0,1,-2)$

Fig. 3. Fuzzy polynomial implications of degree 4 obtained through the vertices of the convex polyhedron given in Theorem 6.

$$\begin{aligned} V_i \in \{ \quad & (0,-1,0,1,0), (0,0,2,0,-1), (0,-1,0,-1,0), (0,1,0,-1,2), (0,1,2,-1,0), (0,1,0,-3,2),\\ & (2,1,0,0,0), \left(\tfrac{4}{3},1,\tfrac{4}{3},-\tfrac{1}{3},-\tfrac{2}{3}\right), (2,1,0,-1,0), (0,1,0,1,-2)\}. \end{aligned}$$

Then I is a fuzzy polynomial implication of degree 4.

Proof. Consider $\alpha, \gamma \geq 0$. Applying Theorem 5, I given by

$$I(x,y) = x(1-y)(\alpha(x-1)^2 + \beta(x-1) + \gamma y^2 + \delta xy + \epsilon y - 1) + 1$$

is a fuzzy polynomial implication of degree 4 if and only if Conditions (1)-(4) and (11)-(13) are satisfied since the other ones are trivially satisfied. Conditions (1)-(4) and (11)-(13), jointly with $\alpha \geq 0$ and $\gamma \geq 0$, constitute a system of linear inequalities whose set of solution defines a convex polyhedron. Applying a vertex enumeration algorithm to this system, the vertices of the convex polyhedron are given by:

$$\begin{aligned} V_i \in \{ \quad & (0,-1,0,1,0), (0,0,2,0,-1), (0,-1,0,-1,0), (0,1,0,-1,2), (0,1,2,-1,0), (0,1,0,-3,2),\\ & (2,1,0,0,0), \left(\tfrac{4}{3},1,\tfrac{4}{3},-\tfrac{1}{3},-\tfrac{2}{3}\right), (2,1,0,-1,0), (0,1,0,1,-2)\}. \quad \square \end{aligned}$$

In Fig. 3, the fuzzy polynomial implications of degree 4 obtained through the vertices of the convex polyhedron are depicted.

Finally, a straightforward corollary is presented which allows to check if a given bivariate polynomial of degree 4 is a fuzzy implication function or not.

Corollary 2. *Let $p(x,y) = \sum_{\substack{0 \leq i,j \leq n \\ i+j \leq 4}} a_{ij} x^i y^j$ be a bivariate polynomial of degree 4 of two variables. Then $p(x,y)$ is a fuzzy polynomial implication of degree 4 if and only if the following conditions hold simultaneously:*

(1) $a_{00} = 1$,
(2) $a_{01} = a_{02} = a_{03} = a_{04} = 0$,
(3) $a_{13} = -a_{10} - a_{11} - a_{12}$,
(4) $a_{22} = -a_{20} - a_{21}$,
(5) $a_{30} = -1 - a_{10} - a_{20}$,
(6) $a_{31} = 1 + a_{10} + a_{20}$,
(7) $a_{40} = 0$.
(8) $a_{11} \geq 0$,

(9) $a_{10} + a_{11} + a_{20} + a_{21} \geq -1$,

(10) $3a_{10} + 2a_{11} + a_{12} \leq 0$,

(11) $2a_{10} + 2a_{11} + a_{12} + a_{20} + a_{21} \leq 1$,

(12) *If* $a_{10} + a_{20} > -1$ *and* $4 \leq \frac{-a_{21}}{1+a_{10}+a_{20}} \leq 6$, *then* $a_{21}^2 - 4a_{11}(1 + a_{10} + a_{20}) \leq 0$,

(13) *If* $a_{10} + a_{20} > -1$ *and* $-2a_{20} \leq a_{21} \leq 2(1 + a_{10})$, *then* $12a_{10}^2 + 4a_{10}(2a_{11} + a_{12} + 3a_{20} + 3) + 4(2a_{11}(a_{20} + 1) + a_{20}(a_{12} + a_{20}) + a_{12}) + 4a_{20}a_{21} + a_{21}^2 \leq 0$,

(14) *If* $a_{10} + a_{11} + a_{12} < 0$ *and* $a_{12} \leq 0 \leq -3(a_{10} + a_{11}) - 2a_{12}$, *then* $3a_{11}(a_{10} + a_{11} + a_{12}) + a_{12}^2 \leq 0$,

(15) *If* $a_{10} + a_{11} + a_{12} < 0$ *and* $a_{12} \leq a_{20} + a_{21} \leq -3(a_{10} + a_{11}) - 2a_{12}$, *then* $3a_{10}^2 + 3a_{10}(2a_{11} + a_{12} + a_{20} + a_{21} + 1) + 3a_{11}^2 + 3a_{11}(a_{12} + a_{20} + a_{21} + 1) + a_{12}^2 + a_{12}a_{20} + a_{12}a_{21} + 3a_{12} + a_{20}^2 + 2a_{20}a_{21} + a_{21}^2 \leq 0$,

(16) *If* $a_{10} + a_{20} > -1$, $a_{10} + a_{11} + a_{12} < 0$, $-3(a_{10} + a_{20} + 1)(a_{10} + a_{11} + a_{12}) - (a_{20} + a_{21})^2 > 0$, $6(a_{10} + a_{20} + 1)(a_{10} + a_{11} + a_{12}) + 2(a_{20} + a_{21})^2 \leq 2a_{12}a_{20} - a_{21}(3(a_{10} + a_{11}) + a_{12}) \leq 0$ *and* $2\big(3(a_{10} + a_{20} + 1)(a_{10} + a_{11} + a_{12}) + (a_{20} + a_{21})^2\big) \leq 2a_{12}(a_{10} + a_{20} + 1) + a_{21}(a_{20} + a_{21}) \leq 0$, *then*

$$\begin{aligned}&12a_{10}^2a_{11} + 4a_{10}(3a_{11}^2 + 3a_{11}(a_{12} + a_{20} + 1) + a_{12}^2) - 3a_{10}a_{21}^2\\&+ 4(3a_{11}^2(a_{20} + 1) + a_{11}(3a_{12}(a_{20} + 1) + a_{20}^2) + a_{12}^2(a_{20} + 1))\\&+ 4a_{20}a_{21}(2a_{11} + a_{12}) + a_{21}^2(a_{11} + a_{12}) \leq 0\end{aligned}$$

(17) *If* $a_{10} + a_{20} > -1$, $a_{10} + a_{11} + a_{12} < 0$,

$$\begin{aligned}&3a_{10}^2 + 3a_{10}(a_{11} + a_{12} + a_{20} + 1) + a_{20}(3(a_{11} + a_{12}) + a_{20})\\&+ 3(a_{11} + a_{12}) + 2a_{20}a_{21} + a_{21}^2 = 0,\end{aligned}$$

$-6a_{20}(a_{10} + a_{11}) - 6a_{10}(a_{10} + a_{11} + 1) - 6a_{11} - 2a_{20}^2 - a_{20}a_{21} + a_{21}^2 = 0$, *and one or more of the following conditions hold:*

(a) $-1 \leq \frac{a_{20}}{a_{20}+a_{21}} \leq 1$,

(b) $-2(a_{10} + a_{20}) \leq a_{21} + 2 \leq 2$,

(c) $-1 \leq \frac{2a_{10}+a_{20}+2}{a_{20}+a_{21}} \leq 1$,

(d) $-2(1 + a_{20}) \leq a_{21} - 2 \leq 2a_{10}$,

then $a_{21}^2 - 4a_{11}(a_{10} + a_{20} + 1) \leq 0$,

(18) $a_{10} \leq 0$,

(19) $2a_{10} + a_{20} \geq -3$,

(20) $2a_{11} + a_{12} + a_{20} + 2a_{21} \leq 3$,

(21) *If* $a_{10} + a_{20} > -1$, *and* $-2a_{20} \leq -a_{20} \leq 3 + 3a_{10} + a_{20}$, *then* $3a_{10}a_{20} + 3a_{10}(a_{10} + 1) + a_{20}^2 \leq 0$,

(22) *If* $a_{10} + a_{20} > -1$, *and* $-2a_{20} \leq a_{21} \leq 3a_{10} + a_{20} + 3$, *then*

$$9a_{10}^2 + 3a_{10}(2a_{11} + a_{12} + 3a_{20} + 3) + 6a_{11}(a_{20} + 1) + 3a_{12}a_{20} + 3a_{12} + 4a_{20}^2 + 4a_{20}a_{21} + a_{21}^2 \leq 0,$$

(23) *If* $a_{10} + a_{11} + a_{12} < 0$, *and* $0 \leq a_{10} + a_{11} \leq -2(a_{10} + a_{11} + a_{12})$, *then* $-3a_{10}^2 - 2a_{10}(a_{11} + 2a_{12}) + a_{11}^2 \leq 0$,

(24) *If* $a_{10} + a_{11} + a_{12} < 0$, *and* $0 \leq a_{10} + a_{11} + 2(a_{20} + a_{21}) \leq -2(a_{10} + a_{11} + a_{12})$, *then*

$$\begin{aligned}&9a_{10}^2 + 2a_{10}(5a_{11} + 4a_{12} + 4a_{20} + 2a_{21} + 6) + a_{11}^2\\&+ 4a_{11}(2a_{20} + a_{21} + 3) + 4(a_{12}(a_{20} + 3) + (a_{20} + a_{21})^2) \leq 0,\end{aligned}$$

(25) *If* $a_{10} + a_{20} > -1$, $a_{10} + a_{11} + a_{12} < 0$,

$$-3a_{10}^2 - 3a_{10}(a_{11} + a_{12} + a_{20} + 1) - a_{20}(3(a_{11} + a_{12}) + a_{20}) - 3(a_{11} + a_{12}) - 2a_{20}a_{21} - a_{21}^2 = 0,$$

$0 \leq a_{21}(a_{10} + a_{11}) - a_{20}(a_{10} + a_{11} + 2a_{12}) \leq -6(a_{10} + a_{20} + 1)(a_{10} + a_{11} + a_{12}) - 2(a_{20} + a_{21})^2$, *and*

$$\begin{aligned}0 &\leq 3a_{10}^2 + 3a_{10}(a_{11} + a_{20} + 1) + 3a_{11}(a_{20} + 1) + 2a_{20}(a_{20} + a_{21})\\&\leq -6(a_{10} + a_{20} + 1)(a_{10} + a_{11} + a_{12}) - 2(a_{20} + a_{21})^2,\end{aligned}$$

then

$$\begin{aligned}&-9a_{10}^3 - 3a_{10}^2(2a_{11} + 4a_{12} + 3a_{20} + 3) + a_{10}(3a_{11}^2 - 6a_{11}(a_{20} + 1)\\&- 4(3a_{12}(a_{20} + 1) + a_{20}^2 + a_{20}a_{21} + a_{21}^2)) + 3a_{11}^2(a_{20} + 1) + 4a_{11}a_{20}a_{21} - 4a_{12}a_{20}^2 \leq 0,\end{aligned}$$

(26) *If* $a_{10}+a_{20}>-1$, $a_{10}+a_{11}+a_{12}<0$,

$$3a_{10}^2+3a_{10}(a_{11}+a_{12}+a_{20}+1)+a_{20}(3(a_{11}+a_{12})+a_{20}) \\ +3(a_{11}+a_{12})+2a_{20}a_{21}+a_{21}^2=0,$$

$-3a_{10}^2-3a_{10}(a_{11}+a_{20}+1)-3a_{11}(a_{20}+1)-2a_{20}(a_{20}+a_{21})=0$, *and one or more of the following conditions hold:*
(a) $-1\le\frac{a_{20}}{a_{20}+a_{21}}\le 0$,
(b) $-a_{20}\le 0\le 3a_{10}+2a_{20}+3$,
(c) $0\le\frac{3a_{10}+2a_{20}+3}{a_{20}+a_{21}}\le 1$,
(d) $-2a_{20}\le a_{21}\le 3a_{10}+a_{20}+3$,
then $3a_{10}a_{20}+3a_{10}(a_{10}+1)+a_{20}^2\le 0$.

(27) $(a_{31},a_{13},a_{22})\neq(0,0,0)$.

Proof. The conditions ranging from (1) to (7) are a direct consequence of Proposition 4.

Using this proposition, the polynomial $p(x,y)$ can be written as a function of the coefficients $a_{10}, a_{11}, a_{12}, a_{20}$ and a_{21}:

$$p(x,y)=1+a_{10}x+a_{20}x^2+a_{11}xy-(1+a_{10}+a_{20})x^3+a_{21}x^2y+a_{12}xy^2-(a_{10}+a_{11}+a_{12})xy^3 \\ -(a_{20}+a_{21})x^2y^2+(1+a_{10}+a_{20})x^3y.$$

It is straightforward to write this polynomial as:

$$p(x,y)=x(1-y)(\alpha(x-1)^2+\beta(x-1)+\gamma y^2+\delta xy+\epsilon y-1)+1,$$

where the coefficients $\alpha,\beta,\gamma,\delta$ and ϵ are the following:

$$\alpha=-1-a_{10}-a_{20},$$
$$\beta=-2-2a_{10}-a_{20},$$
$$\gamma=a_{10}+a_{11}+a_{12},$$
$$\delta=a_{20}+a_{21},$$
$$\epsilon=a_{10}+a_{11}.$$

So, we can apply Theorem 5 and the conditions ranging from (8) to (26) are a consequence of this theorem. Finally, Condition (27) ensures that the fuzzy polynomial implication is of degree 4 since $a_{40}=a_{04}=0$ by Conditions (2) and (7). □

5. Conclusions and future work

In this paper, an in-depth study of those fuzzy implication functions whose expression is given by a polynomial of two variables is presented. The interest on this family of fuzzy implication function relies on the simplicity of their expression, their continuity and their lower propagation of errors with respect to other expressions. These facts make them excellent candidates for their use in any application.

First of all, several results are proved for fuzzy polynomial implications of any degree. These results ensure the fulfilment of the required properties on the boundary of the domain $[0,1]^2$. However, the complexity of polynomials of any degree makes impossible to study the monotonicities for any degree n. Therefore, in Section 4 the main results of the paper are included. Indeed, characterisation results for all fuzzy polynomial implications of degree ≤ 4 are achieved and analysed. These results prove the increasing complexity of the study with respect to the degree of the fuzzy polynomial implication, needing 19 conditions on the coefficients to characterise degree 4.

As future work, we want to study which additional properties are satisfied by the fuzzy polynomial implications of degree 4. Furthermore, it would be worthy of interest to investigate if there exist any construction method of fuzzy polynomial implications of higher degrees from fuzzy polynomial implications of low degrees which provides the complete family of these operators.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

This paper is part of the R&D&I project PID2020-113870GB-I00- "Desarrollo de herramientas de Soft Computing para la Ayuda al Diagnóstico Clínico y a la Gestión de Emergencias (HESOCODICE)", funded by MCIN/AEI/10.13039/501100011033/.

References

[1] C. Alsina, M. Frank, B. Schweizer, Associative Functions. Triangular Norms and Copulas, World Scientific Publishing Company, Singapore, 2006.
[2] M. Baczyński, B. Jayaram, On the characterization of (S,N)-implications, Fuzzy Sets Syst. 158 (2007) 1713–1727.
[3] M. Baczyński, B. Jayaram, Fuzzy Implications, Studies in Fuzziness and Soft Computing, vol. 231, Springer, Berlin Heidelberg, 2008.
[4] M. Baczynski, B. Jayaram, S. Massanet, J. Torrens, Fuzzy implications: past, present, and future, in: J. Kacprzyk, W. Pedrycz (Eds.), Springer Handbook of Computational Intelligence, Springer Berlin Heidelberg, Berlin, Heidelberg, 2015, pp. 183–202.
[5] H. Bustince, J. Fernandez, J. Sanz, M. Baczyński, R. Mesiar, Construction of strong equality index from implication operators, Fuzzy Sets Syst. 211 (2013) 15–33.
[6] J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Dordrecht, 1994.
[7] H. Hamacher, Über logische Aggregationen nicht binär explizierter Entscheidungskriterien, chapter Ein axiomatischer, Beitrag zur Normativen Entscheidungstheorie, Verlag, Frankfurt, 1978.
[8] S. Massanet, Tidying up the mess of classes of fuzzy implication functions, in: R. Halas, M. Gagolewski, R. Mesiar (Eds.), Book of Abstracts of the 10th International Summer School on Aggregation Operators, Olomouc University, 2019, pp. 1–2.
[9] S. Massanet, J.V. Riera, D. Ruiz-Aguilera, On fuzzy polynomial implications, in: A. Laurent, O. Strauss, B. Bouchon-Meunier, R.R. Yager (Eds.), Information Processing and Management of Uncertainty in Knowledge-Based Systems, Springer International Publishing, Cham, 2014, pp. 138–147.
[10] S. Massanet, J.V. Riera, D. Ruiz-Aguilera, On (op)-polynomial implications, in: J.M. Alonso, H. Bustince, M. Reformat (Eds.), Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (IFSA-EUSFLAT-15), Gijón, Spain., June 30, 2015, Atlantis Press, 2015.
[11] S. Massanet, J.V. Riera, D. Ruiz-Aguilera, On rational fuzzy implication functions, in: 2016 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2016, pp. 272–279.
[12] S. Massanet, J. Torrens, On the characterization of Yager's implications, Inf. Sci. 201 (2012) 1–18.