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Modus tollens with respect to uninorms: U-Modus Tollens

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ABSTRACT

In fuzzy logic and approximate reasoning the inference rule given by the Modus Tollens usually derives into an inequality involving three logical operators: a conjunction, an implication function and a negation. Until now, in this scenario the conjunction has been commonly modeled by a t-norm, but recently the possibility of using a more general conjunction has been pointed out. In this work, we want to generalize the Modus Tollens inequality by using a conjunctive uninorm instead of a t-norm, leading to the so-called U-Modus Tollens. First, we give a study of this new property for implication functions in general and then we specially focus on residual implications derived from uninorms. In all cases, we prove that there are a lot of solutions of the U-Modus Tollens and we give a characterization of all the solutions in some particular cases.

1. Introduction

Implication functions are commonly used not only to model fuzzy conditionals, but also to carry out all inference processes in any fuzzy rule based system. Thus, the importance of implication functions mainly lies in their applications, which cover a wide range of different fields, in particular, all those where fuzzy control and approximate reasoning can be applied (see for instance [3,5,18]). This is also the main reason for which it is important to have a wide range of implication functions to be used in different contexts (see [29]).

From the theoretical point of view, one of the main interesting topics on implication functions is the study and the characterization of those implications that satisfy certain additional properties that usually arise from the concrete applications. There are many of these properties, but we can highlight among them those derived from the basic inference rules: Modus Ponens and Modus Tollens (see [9,10,12]). In the framework of fuzzy logic, these inference rules are guaranteed when the involved logical operators satisfy the following functional inequalities, respectively:

$$T(x, I(x, y)) \le y \quad \text{for all} \quad x, y \in [0, 1], \tag{1}$$

and

$$T(N(y), I(x, y)) \le N(x) \quad \text{for all} \quad x, y \in [0, 1], \tag{2}$$

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where T is a t-norm, I is an implication function and N is a negation.

When implication functions have to be used in managing inference processes, one of these two properties or both of them are essential. Thus, due to their significance, both inequalities have been largely studied in the literature for many authors (see for instance [2,3,16,18,26–28,30]). First studies were carried out for implication functions derived from t-norms and t-conorms. Specifically, residuated implication functions and (S,N)-implications were studied in [2,26,27], whereas the case of QL and D-implications was studied in [28]. As a generalization of these classes of implications, their counterparts derived from more general kinds of aggregation functions have been introduced, specially those derived from uninorms like RU-implications and (U,N)-implications (see for instance [1,4,6,17,22–24]). Recently, both the Modus Ponens and the Modus Tollens have been studied also for these two kinds of implications derived from uninorms, (see [16] and [15], respectively).

In fact, although uninorms were initially introduced in the framework of aggregation functions (see [8,31]), they have also been studied and used as logical operators due to the fact that they are always conjunctive (their value in the point $(0,1)$ is 0) or disjunctive (their value in $(0,1)$ is 1). In particular, conjunctive uninorms are commonly used as fuzzy conjunctions. Thus, the idea of substituting the t-norm T by a conjunctive uninorm U in the Modus Ponens and the Modus Tollens inference rules becomes natural and it turns out to be an interesting line of research. Note that, in the Modus Ponens case, this idea was already carried out in [19] leading to the so-called U-Modus Ponens (or equivalently U-conditionality). Initial results in these works prove that the usual implications derived from t-norms and t-conorms, as well as generated Yager implications, never satisfy U-Modus Ponens, whereas implications derived from uninorms appear to be the most suitable ones to satisfy it. In this direction, the case of RU-implications has already been solved in [19], and the case of (U,N)-implications has been recently studied in [20].

On the contrary, a similar study for the case of the Modus Tollens is still missing. In this paper, we want to deal precisely with this task and we want to study the same generalization of the Modus Tollens. That is, we want to study the property obtained from Equation (2) by substituting the t-norm T by a conjunctive uninorm U, that we will call the U-Modus Tollens:

$$U(N(y), I(x,y)) \leq N(x) \quad \text{for all} \quad x, y \in [0,1], \tag{3}$$

where U is a conjunctive uninorm, I is an implication function and N is a fuzzy negation. We want to highlight that in uninorm-based fuzzy logics (see [14]) a proposition P is considered true when its assigned value is greater than or equal to e (where e denotes the neutral element of uninorm U). In this context, the rule given by Equation (3) could be interpreted as follows: "If the value of *not* Q is greater than or equal to e and the value of $P \to Q$ is greater than or equal to e, then the value of *not* P is also greater than or equal to e".

This paper is organized as follows. After this introduction, Section 2 is devoted to some preliminaries compiling all the notations and concepts necessary to follow the article. In Section 3 the U-Modus Tollens is introduced and it is analyzed for general implication functions. Section 4 is devoted to the case of RU-implications derived from the three most usual classes of conjunctive uninorms: uninorms in $\mathcal{U}_{\min}$, representable uninorms and idempotent uninorms. Finally, Section 5 includes some conclusions and some future work proposals.

2. Preliminaries

We recall some concepts that we will use along the paper.

Definition 2.1. ([11]) A function $N : [0,1] \to [0,1]$ is said to be a *fuzzy negation* if it is decreasing with $N(0) = 1$ and $N(0) = 1$. A fuzzy negation N is said to be

- *Strict* when it is strictly decreasing and continuous.
- *Strong* when it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Definition 2.2. ([3,7]) A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be an *implication function*, or an *implication*, if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.
(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.
(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that it follows from the definition, that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition.

Special interesting properties for implication functions are:

- The *neutrality property*,

$$I(1,y) = y \text{ for all } y \in [0,1]. \tag{NP}$$

- The *identity principle*,

$$I(x,x) = 1 \text{ for all } x \in [0,1]. \tag{IP}$$

- The *contrapositive property* with respect to a negation N

$$I(x,y) = I(N(y),N(x)) \text{ for all } x,y \in [0,1]. \tag{$CP(N)$}$$

Definition 2.3. ([3,7]) Let I be a fuzzy implication function. The function N_I given by $N_I(x) = I(x,0)$ for all $x \in [0,1]$ is always a fuzzy negation, which is called the *natural negation* of I.

Definition 2.4. ([8,31]) A *uninorm* is a two-place function $U : [0,1]^2 \longrightarrow [0,1]$ which is associative, commutative, increasing in each place, and such that there exists some element $e \in [0,1]$, called *neutral element*, such that $U(e,x) = x$ for all $x \in [0,1]$.

It is clear that uninorms generalize both t-norms and t-conorms, since they are retrieved from uninorms just taking $e = 1$ and $e = 0$, respectively. For any other value $e \in]0,1[$, the operation works as a t-norm in the square $[0,e]^2$, as a t-conorm in $[e,1]^2$, and its values are between minimum and maximum in the set of points $A(e)$ given by

$$A(e) = [0,e[\times]e,1] \cup]e,1] \times [0,e[.$$

We will usually denote a uninorm with neutral element e and underlying t-norm T and t-conorm S, by $U \equiv \langle T,e,S\rangle$. Moreover, for a uninorm U, it is always $U(1,0) \in \{0,1\}$ and U is said to be *conjunctive* when $U(1,0) = 0$, and *disjunctive* when $U(1,0) = 1$.

There are many different kinds of uninorms. The most common ones can be found in [13]. Let us recall here the structure of the three most-used classes of conjunctive uninorms that will be used along the paper.

Proposition 2.1. ([8,13]) *Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$. If $U(0,1) = 0$, then the function $x \mapsto U(x,1)$ is continuous except for $x = e$ if and only if U is given by*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{if } (x,y) \in A(e), \end{cases}$$

where T is a t-norm and S is a t-conorm.

We will denote by $\mathcal{U}_{\min}$ the family of these uninorms and by $U \equiv \langle T,e,S\rangle_{\min}$ the uninorm in $\mathcal{U}_{\min}$ with neutral element e and underlying t-norm T and t-conorm S.

Idempotent uninorms were completely characterized for the general case in [25] in the following way.

Proposition 2.2. ([25,13]) *A binary operator $U : [0,1]^2 \to [0,1]$ is an idempotent uninorm with neutral element $e \in [0,1]$ if and only if there exists a decreasing function $g : [0,1] \to [0,1]$, symmetric with respect to the main diagonal, with $g(e) = e$, such that*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g^2(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g^2(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g^2(x), \end{cases}$$

where $g^2(x) = g(g(x))$, being U commutative in the points (x,y) such that $y = g(x)$ with $x = g^2(x)$.

Any idempotent uninorm U with neutral element e and associated function g will be denoted by $U \equiv \langle g,e\rangle_{\text{ide}}$, and the class of idempotent uninorms will be denoted by $\mathcal{U}_{\text{ide}}$. Obviously, for any of these uninorms the underlying t-norm T_U is the minimum and the underlying t-conorm S_U is the maximum.

Another well known class of uninorms is the one given by all uninorms that can be represented by unary functions.

Definition 2.5. ([8,13]) A uninorm U with neutral element $e \in]0,1[$ is said to be representable if there exists a strictly increasing and continuous function $h : [0,1] \to [-\infty,+\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that U is given by

$$U(x,y) = \begin{cases} h^{-1}(h(x)+h(y)) & \text{if } (x,y) \notin \{(0,1),(1,0)\}, \\ 0 \text{ (or 1)} & \text{otherwise.} \end{cases}$$

Function h is called an additive generator of U. Thus it is clear that there are two different representable uninorms with the same generator h, a conjunctive one (which is then left-continuous) and a disjunctive one (which is right-continuous). Any representable uninorm U with neutral element e and additive generator h will be denoted by $U \equiv \langle e, h\rangle_{\text{rep}}$, and the class of representable uninorms will be denoted by $\mathcal{U}_{\text{rep}}$. Moreover, for any representable uninorm $U \equiv \langle e, h\rangle_{\text{rep}}$ the function

$$N_h(x) = h^{-1}(-h(x)) \quad \text{for all } x \in [0,1]$$

is always a strong negation associated to the uninorm U.

On the other hand, different classes of implications derived from uninorms have also been studied. Among them, only RU-implications will be used along the paper and for this reason they are recalled here.

Definition 2.6. ([6]) Let U be a uninorm. The *residual operation* derived from U is the binary operation given by

$$I_U(x, y) = \sup\{z \in [0,1] \mid U(x,z) \le y\} \tag{4}$$

for all $x, y \in [0,1]$.

Proposition 2.3. ([6]) *Let U be a uninorm and I_U its residual operation. Then I_U is an implication, called RU-implication, if and only if the following condition holds*

$$U(x,0) = 0 \quad \textit{for all} \quad x < 1. \tag{5}$$

This property includes all conjunctive uninorms but also many disjunctive ones, for instance the classes of representable and idempotent uninorms with $g(0) = 1$ (see [6] and [22]). However, when we deal with left-continuous uninorms U, we clearly have that U satisfies condition (5) if and only if it is conjunctive. There are some properties of RU-implications derived from uninorm, that can be deduced directly from the definition, or can be proved in the similar way that it was done for those derived from t-norms. For more details see [6].

3. Modus tollens with respect to uninorms: *U*-Modus Tollens

As we have already mentioned, the main objective of this work is the study of the generalization of the Modus Tollens property obtained by changing the t-norm T by a uninorm U, in a similar way as it was done for the Modus Ponens in [19]. For obvious reasons we will refer to this generalization as the *U-Modus Tollens*. Thus, let us begin by giving the formal definition in the framework of fuzzy logic.

Definition 3.1. Let I be a fuzzy implication function, U a uninorm and N a fuzzy negation. It is said that I and N satisfy the *U-Modus Tollens* property with respect to U when

$$U(N(y), I(x,y)) \le N(x) \quad \text{for all} \quad x, y \in [0,1]. \tag{6}$$

Note that the definition has been given for a uninorm in general and not for a conjunctive one.[1] This is because this property on U is directly derived from the definition of U-Modus Tollens as it is proved in the following proposition.

Proposition 3.1. *Let I be a fuzzy implication function and N a negation that satisfy the U-Modus Tollens with respect to a uninorm U. Then the uninorm U must necessarily be conjunctive.*

Proof. If I and N satisfy the U-Modus Tollens with respect to U, just take $x = 1$ and $y = 0$ in Equation (6) to obtain $U(N(0), I(1,0)) \le N(1)$, that is, $U(1,0) = 0$. □

In view of the proposition above, from now on, we will consider only conjunctive uninorms in the U-Modus Tollens property. By Definition 3.1, in general, U-Modus Tollens depends on the uninorm U, the fuzzy implication I, and the fuzzy negation N. Example 3.1 shows that, for any conjunctive uninorm U, if we require U-Modus Tollens to be valid for some particular negation N, the admissible implications I depend on the choice of N.

Example 3.1. Let us see how the U-Modus Tollens behaves in the cases where the extreme fuzzy negations are involved, that is, when we consider the smallest and largest fuzzy negations.

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[1] Recall that a t-norm T is used to model a conjunction in fuzzy logic. Thus, the normal generalization would be to take a conjunctive uninorm.

i) Let us consider the smallest fuzzy negation $N = N_{D_1}$, given by

$$N_{D_1}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Note that, if we take any conjunctive uninorm U and the negation N_{D_1}, any implication function I trivially satisfies Equation (6) for all $y > 0$ (because then $N_{D_1}(y) = 0$) and also when $x = 0$ (because then $N_{D_1}(x) = 1$). On the other hand, when $y = 0$ and $x > 0$ Equation (6) holds if and only if

$$U(N(0), I(x, 0)) = U(1, N_I(x)) \leq N(x) = 0,$$

which implies $N_I(x) = 0$ for all $x > 0$. Thus, we get the following conclusion: "*An implication function I satisfies the U-Modus Tollens with respect to a conjunctive uninorm U and the fuzzy negation N_{D_1} if and only if* $N_I = N_{D_1}$".

ii) Let us consider now the greatest fuzzy negation $N = N_{D_2}$, given by

$$N_{D_2}(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

A similar argument shows the following conclusion: "An implication function I fulfills the U-Modus Tollens with respect to a conjunctive uninorm U and the fuzzy negation N_{D_2} if and only if $I(1, y) = 0$ for all $y < 1$".

Similarly to the case of t-norms, when the uninorm U is left-continuous, we can easily derive a general result about U-Modus Tollens as follows.

Proposition 3.2. *Let U be a conjunctive left-continuous uninorm, I_U its residual implication function and N a fuzzy negation. Then, an implication function I satisfies the U-Modus Tollens with respect to U and N if and only if*

$$I(x, y) \leq I_U(N(y), N(x)) \quad \textit{for all } x, y \in [0, 1],$$

Proof. Just recall that when U is left-continuous the residuation property holds (see [6]) and then

$$U(N(y), I(x, y)) \leq N(x) \iff I(x, y) \leq I_U(N(y), N(x)). \quad \square$$

However, the above condition does not hold when U is not left-continuous and moreover, even when it holds, it can be difficult to check it in many cases. Thus, we want to deeply study the U-Modus Tollens condition to give more specific results. A first analysis of the U-Modus Tollens allows us to give some general results that we list in the following proposition.

Proposition 3.3. *Let I be a fuzzy implication function and N a negation that satisfy the U-Modus Tollens with respect to a conjunctive uninorm U. Let e be the neutral element of U, then the following properties are satisfied*

1. $U(N(y), I(1, y)) = 0$ *for all* $y \in [0, 1]$.
2. $N_I(x) \leq N(x)$ *for all* $x \in [0, 1]$. *Also, for all* $x \in [0, 1]$ *such that* $N_I(x) \geq e$ *it must be* $N(x) = 1$.
3. *Take* $\alpha_N = \sup\{x \in [0, 1] \mid N(x) \geq e\}$. *Then*
   a) *If* $\alpha_N = 0$ *then* $N(x) < e$ *for all* $x > 0$.
   b) *If* $\alpha_N > 0$ *then it must be* $I(1, y) = 0$ *for all* $y < \alpha_N$.
4. *If I satisfies (NP), then* $\alpha_N = 0$.
5. *If I satisfies (IP) and N is non filling ($N(x) < 1$ for $x > 0$), it must be* $\alpha_N = 0$.

Proof. Let us prove them step by step.

1. This first item is a direct consequence of the U-Modus Tollens property, just taking $x = 1$ in Equation (6).
2. Consider $N_I(x) = I(x, 0)$, then we have

$$N_I(x) = U(e, I(x, 0)) \leq U(1, I(x, 0)) \leq N(x),$$

where the last inequality follows by taking $y = 0$ in Equation (6).
On the other hand, if $N_I(x) \geq e$, taking again $y = 0$ in Equation (6), we obtain

$$N(x) \geq U(N(0), I(x, 0)) = U(1, N_I(x)) \geq U(1, e) = 1.$$

3. Let us consider now $\alpha_N = \sup\{x \in [0, 1] \mid N(x) \geq e\}$.

a) If $\alpha_N = 0$, we directly derive from the definition of α_N that $N(x) < e$ for all $x > 0$.
b) If $\alpha_N > 0$ we get $N(y) \geq e$ for all $y < \alpha_N$ and consequently,

$$I(1, y) = U(e, I(1, y)) \leq U(N(y), I(1, y)) \leq N(1) = 0,$$

that is, $I(1, y) = 0$ for all $y < \alpha_N$.

4. If $\alpha_N > 0$, we get $I(1, y) = 0$ for all $y < \alpha_N$ and consequently I does not satisfy (NP).
5. Suppose in this case that $I(x, x) = 1$ for all $x \in [0, 1]$ and $N(x) < 1$ for $x > 0$. In this case, suppose again that there is some $x > 0$ with $N(x) \geq e$. Taking $x = y$ in Expression (6) we will get

$$N(x) \geq U(N(x), I(x, x)) \geq U(e, 1) = 1,$$

which is a contradiction. Thus, it must be again $N(x) < e$ for all $x > 0$ and consequently, $\alpha_N = 0$. □

Remark 3.1. Some interesting consequences can be derived from Proposition 3.3 as follows.

(i) If we want to deal with continuous fuzzy negations in the U-Modus Ponens, then we will have $\alpha_N > 0$. From Item 4 in Proposition 3.3 we get that any possible solution of the U-Modus Ponens fails to satisfy (NP). Consequently, all usual implications derived from t-norms and t-conorms (that is, R, (S, N), QL and D-implications), as well as Yager generated implications, never satisfy U-Modus Tollens (since all of them satisfy (NP)). Fortunately, we will see in next sections that many solutions can be found among those implications derived from uninorms, especially among RU-implications.
(ii) On the other hand, if we want to consider implication functions satisfying (NP), the same argument leads to the necessity of considering negations with $\alpha_N = 0$, that is, negations N such that $N(x) < e$ for all $x > 0$ and consequently, non-continuous.

In the rest of this section we will deal with the case $\alpha_N = 0$ and also when the involved implications satisfy (NP). We will characterize in particular all solutions in the case when the negation N is strictly decreasing in the interval $]0, e[$ and U is a uninorm in $\mathcal{U}_{\min}$. We start by proving that for this kind of uninorms, U-Modus Tollens always holds for all $x \leq y$ and so it is sufficient to prove it for (x, y) such that $y < x$.

Proposition 3.4. *Let I be an implication function, N a fuzzy negation with $\alpha_N = 0$, and U a uninorm in $\mathcal{U}_{\min}$ with neutral element $e \in]0, 1[$. Then I, N satisfy the U-Modus Tollens with respect to U if and only if*

$$U(N(y), I(x, y)) \leq N(x) \quad \text{for all } y < x.$$

Proof. We only need to prove that the U-Modus Tollens is always satisfied in points where $x \leq y$. Since it is obvious when $x = 0$, we deal with values $0 < x \leq y$. In this case we have $N(y) \leq N(x)$ and since $\alpha_N = 0$, it is $N(x) < e$ for all $x > 0$. Then we have two cases:

a) If $I(x, y) > e$, then

$$U(N(y), I(x, y)) \leq U(N(x), I(x, y)) = \min\{N(x), I(x, y)\} = N(x).$$

b) If $I(x, y) \leq e$, then

$$U(N(y), I(x, y)) \leq U(N(x), I(x, y)) = eT_U\left(\frac{N(x)}{e}, \frac{I(x, y)}{e}\right) \leq e\,\frac{N(x)}{e} = N(x). \quad \square$$

We now study the case in which the implication I satisfies (NP), which is the case of most of the implications derived from t-norms and t-conorms.

Proposition 3.5. *Let I be an implication function that satisfies (NP), N a fuzzy negation, and U a conjunctive uninorm with neutral element $e \in]0, 1[$. If I, N satisfy the U-Modus Tollens with respect to U, then:*

1. *$U(N(y), y) = 0$ for all $y \in [0, 1]$.*
2. *$N(x) = 0$ for all $x \geq e$ and $N(x) < e$ for all $x > 0$.*
3. *$U(N(y), I(e, y)) = 0$ for all $y \in [0, 1]$.*
4. *If N is strictly decreasing in the interval $]0, e[$, then $I(x, y) < e$ for all $y < x < e$.*

Proof. Let us prove all items one by one.

1. $U(N(y), y) = 0$ for all $y \in [0, 1]$ follows directly by taking $x = 1$ in Equation (6) since I satisfies (NP).

2. According to Proposition 3.3 we know that if I satisfies (NP), then $\alpha_N = 0$ and $N(x) < e$ for all $x > 0$. Moreover, just by putting $y = e$ in the previous condition, we obtain $N(e) = U(N(e), e) = 0$ and consequently, $N(x) = 0$ for all $x \geq e$, by decreasingness.
3. Now, taking $x = e$ in Equation (6) we get $U(N(y), I(e, y)) \leq N(e) = 0$, and so $U(N(y), I(e, y)) = 0$ for all $y \in [0, 1]$.
4. To prove this item let us consider some x, y such that $y < x < e$ and suppose that $I(x, y) > e$. First, note that $y > 0$ because $I(x, 0) = N_I(x) \leq N(x) < e$. Consequently, we have $N(y) < e$ and so, since N is strictly decreasing in $]0, e[$,

   $$U(N(y), I(x, y)) \geq \min\{N(y), I(x, y)\} = N(y) > N(x),$$

   leading to a contradiction. Thus, we obtain $I(x, y) < e$ for all $y < x < e$. $\square$

When N is a negation strictly decreasing in the interval $]0, e[$ and the implication I satisfies (NP), we can give a characterization of those uninorms in $\mathcal{U}_{\min}$ that satisfy the U-Modus Tollens, as follows.

Theorem 3.1. *Let I be an implication function that satisfies (NP), $U \equiv \langle T_U, e, S_U \rangle_{\min}$ a uninorm in $\mathcal{U}_{\min}$ with neutral element $e \in]0, 1[$, and N a fuzzy negation strictly decreasing in $]0, e[$. Then I, N satisfy the U-Modus Tollens with respect to U if and only if the following properties hold:*

1. *$U(N(y), I(e, y)) = 0$ for all $y \in [0, 1]$.*
2. *$N(x) = 0$ for all $x \geq e$ and $N(x) < e$ for all $x > 0$.*
3. *$I(x, y) < e$ for all $y < x < e$.*
4. *I' and N' verify the Modus Tollens with respect to T_U for all $y < x$, where I' and N' are given by*

$$I'(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{I(ex, ey)}{e} & \text{if } y < x. \end{cases} \tag{7}$$

$$N'(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{N(ex)}{e} & \text{otherwise}. \end{cases} \tag{8}$$

Proof. Let us suppose first that I and N satisfy the U-Modus Tollens with respect to U. Properties 1, 2 and 3 hold from Proposition 3.5 and thus, we only need to prove that I' and N' satisfy the Modus Tollens with respect to T_U for all $y < x$. We will distinguish two cases:

- If $y = 0$ then,

  $$T_U(N'(y), I'(x, 0)) = T_U\left(1, \frac{I(ex, 0)}{e}\right) = \frac{I(ex, 0)}{e} = \frac{N_I(ex)}{e} \leq \frac{N(ex)}{e} = N'(x),$$

  where the last inequality is due to Proposition 3.3.
- If $y > 0$, we have

  $$T_U(N'(y), I'(x, y)) = T_U\left(\frac{N(ey)}{e}, \frac{I(ex, ey)}{e}\right) = \frac{U(N(ey), I(ex, ey))}{e} \leq \frac{N(ex)}{e} = N'(x),$$

  where the last inequality holds because I, N satisfy U-Modus Tollens with respect to U.

Conversely, suppose that conditions 1-4 hold and we will prove that I, N satisfy the U-Modus Tollens with respect to U. According to Proposition 3.4 the U-Modus Tollens needs to be checked only in points (x, y) where $y < x$. Thus, we will consider three cases:

- If $y = 0$, then

  $$U(N(y), I(x, y)) = U(1, I(x, 0)) = U(1, N_I(x)) = N_I(x),$$

  where the last equality holds because $U \in \mathcal{U}_{\min}$. Then the result follows from Proposition 3.3.
- If $0 < y < x < e$ then $I(x, y) < e$ and $N(y) < e$ by conditions 3 and 2, respectively. In this case,

  $$U(N(y), I(x, y)) = e \cdot T_U\left(\frac{N(y)}{e}, \frac{I(x, y)}{e}\right) = e \cdot T_U\left(N'\left(\frac{y}{e}\right), I'\left(\frac{x}{e}, \frac{y}{e}\right)\right) \leq e \cdot N'\left(\frac{x}{e}\right) = N(x),$$

  where the last inequality holds from condition 4.

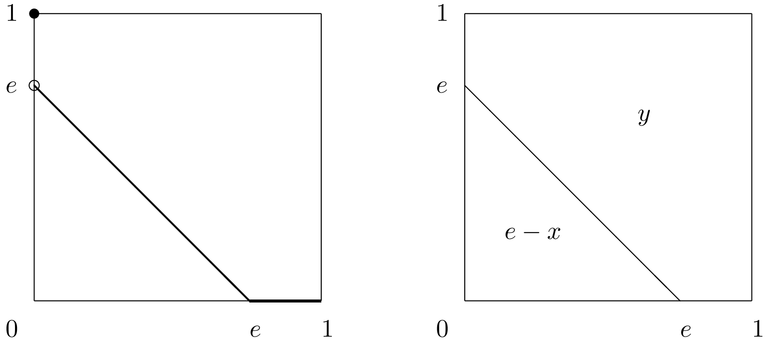

Fig. 1. Plot of the fuzzy negation N_e (left) and structure of the fuzzy implication I_e (right) given in Example 3.2.

- If $x \geq e$, then $I(x, y) \leq I(e, y)$ and hence,

$$U(N(y), I(x, y)) \leq U(N(y), I(e, y)) = 0,$$

where in this case the last equality is due to condition 1. □

Remark 3.2. Let us point out two interesting facts.

i) According to the theorem above, the proof that I, N satisfy the U-Modus Tollens with respect to a uninorm $U \equiv \langle T_U, e, S_U\rangle_{\min}$ in $\mathcal{U}_{\min}$, requires in particular to check that I' and N' satisfy the Modus Tollens with respect to T_U. However, this question was already solved in [26,27] and consequently, our characterization theorem is complete. In fact, many examples can be derived from conditions in Theorem 3.1 (see Example 3.2 below).
ii) Note that in Theorem 3.1, the t-conorm S_U plays no role in the satisfaction of the U-Modus Tollens and so, S_U can be any t-conorm. Moreover, the same property holds when the values of the implication I are out of the rectangle $[0, e[^2$, with the unique condition that it satisfies (NP).

Example 3.2. Let us consider the uninorm in $\mathcal{U}_{\min}$, $U \equiv \langle T_U, e, S_U\rangle_{\min}$ where $T_U = T_{\mathbf{LK}}$ is the Łukasiewicz t-norm and S_U is any t-conorm. Let N_e be the negation given by

$$N_e(x) = \begin{cases} 1 & \text{if } x = 0, \\ e - x & \text{if } 0 < x < e, \\ 0 & \text{if } x \geq e. \end{cases}$$

On the other hand, we consider the following fuzzy implication

$$I_e(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ \max(e - x, y) & \text{if } 0 < x \leq e \text{ and } 0 \leq y \leq e, \\ y & \text{otherwise}. \end{cases}$$

It is easy to see that these operators satisfy all the conditions of Theorem 3.1 and therefore I_e, N_e satisfy the U-Modus Tollens with respect to the uninorm U (Fig. 1).

4. Modus Tollens for *RU*-implications

In this section, given a conjunctive uninorm U and a fuzzy negation N, we will investigate which RU-implications satisfy the U-Modus Tollens inequality. We will do it for RU-implications derived from the three classes of uninorms recalled in the preliminaries. Note that in the previous section we have seen that, for fuzzy negations N with $\alpha_N > 0$ (case that includes all continuous negations), only implications I such that $I(1, y) = 0$ for all $y < \alpha_N$ can satisfy the U-Modus Tollens. Fortunately, this condition is satisfied by RU-implications derived from representable uninorms and for some idempotents ones and so, in these cases we will see that continuous (even strict and strong) negations can be considered.

We will study the U-Modus Tollens for RU-implications derived from uninorms in the class of $\mathcal{U}_{\min}$, from representable uninorms and from idempotent uninorms, and we will devote a subsection to each of these cases.

4.1. Case of RU-implications derived from uninorms in $\mathcal{U}_{\min}$

In this subsection, we will deal with RU-implications derived from uninorms in $\mathcal{U}_{\min}$, that is, uninorms $U_0 \equiv \langle T_0, e_0, S_0\rangle_{\min}$ with neutral element $e_0 \in]0,1[$ and underlying t-norm and t-conorm T_0 and S_0, respectively. Recall that for this kind of uninorms, RU-implications are given by the following structure.

Proposition 4.1. (Theorem 5.4.7 in [3]) *Let $U \equiv \langle T_U, e, S_U\rangle_{\min}$ be a uninorm in $\mathcal{U}_{\min}$ and I_U its residual implication. Then*

$$I_U(x,y) = \begin{cases} 1 & \text{if } x \le y < e, \\ eI_{T_U}(\frac{x}{e}, \frac{y}{e}) & \text{if } y < x \le e, \\ e + (1-e)I_{S_U}(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } e \le x \le y, \\ e & \text{if } e \le y < x, \\ y & \text{otherwise.} \end{cases} \tag{9}$$

For this kind of RU-implications we again get that the U-Modus Tollens can be satisfied only with negations N with $\alpha_N = 0$ and consequently, with non-continuous negations. Moreover, for this kind of RU-implications we have the following result.

Proposition 4.2. *Let I_{U_0} be a fuzzy implication derived from $U_0 \equiv \langle T_0, e_0, S_0\rangle_{\min}$, N a fuzzy negation, and $U \equiv \langle T_U, e, S_U\rangle$ a conjunctive uninorm. If I_{U_0}, N satisfy the U-Modus Tollens with respect to U, then the following properties hold:*

1. $U(N(y), y) = 0$ *for all* $y \le e_0$.
2. $U(N(y), 1) = N(y)$ *for all* $y < e_0$.
3. $\alpha_N = 0$ *(that is,* $N(x) < e$ *for all* $x > 0$*).*
4. *If* $e \le e_0$ $N(x) = 0$ *for all* $x \ge e$.
5. *If* $e_0 < e$*,* T_U *is a continuous t-norm and* $U(e_0, e_0) = e_0$*, then* $N(x) < e_0$ *for all* $x > 0$ *and* $N(x) = 0$ *for all* $x \ge e_0$.

Proof. 1. Follows directly from Equation (6) by putting $x = 1$ and taking into account that $I_{U_0}(1, y) = y$ for all $y \le e_0$.

2. For any $x < e_0$, we have $I_{U_0}(x, x) = 1$ and if we take $x = y$ in Equation (6), we have

$$U(N(y), I_{U_0}(y,y)) = U(N(y), 1) \le N(y).$$

The result follows from the fact that $U(N(y), 1) \ge N(y)$ holds always.

3. It is a direct consequence of Proposition 3.3, because it is $I_{U_0}(1, y) > 0$ for all $y > 0$.
4. If we take $y = e$ in Property 1, we have $U(N(e), e) = N(e) = 0$ and by decreasingness, $N(x) = 0$ for all $x \ge e$.
5. First, let us see that $N(x) < e_0$ for all $0 < x < e_0$. Indeed, since T_U is continuous and $e_0 < e$ is an idempotent element, if we have that $N(x) \ge e_0$ with $e_0 > x > 0$, then

$$U(e_0, x) = x \qquad \text{for all} \qquad 0 \le x \le e_0 \tag{10}$$

and then, applying Property 1 we get

$$0 = U(N(x), x) \ge U(e_0, x) = x$$

leading to a contradiction. Now, we have that $N(x) < e_0$ for all $x > 0$ by decreasingness, and we only need to prove that $N(e_0) = 0$ to conclude the proof. But this is clear because $N(e_0) < e_0$ and then

$$0 = U(N(e_0), e_0) = N(e_0),$$

where the first equality is due to Property 1 and the second is due to Equation (10). □

In order to characterize the solutions in this case, we will distinguish three different possibilities according to the order of the neutral elements e and e_0. We begin by the most simple case when both elements coincide.

Theorem 4.1. *Let I_{U_0} be the RU-implication derived from $U_0 \equiv \langle T_0, e_0, S_0\rangle_{\min}$, N a fuzzy negation, and $U \equiv \langle T_U, e, S_U\rangle$ a conjunctive uninorm with neutral element $e = e_0$. Then I_{U_0} and N satisfy the U-Modus Tollens with respect to U, if and only if the following properties hold*

1. $U(N(y), y) = 0$ *for all* $y \le e$*, and* $U(N(y), 1) = N(y)$ *for all* $y < e$.
2. $N(x) < e$ *for all* $x > 0$*, and* $N(x) = 0$ *for all* $x \ge e$.
3. I_{T_0} *and* N' *satisfy the Modus Tollens with respect to the t-norm* T_U *for all* $y < x$*, where* I_{T_0} *is the residual implication derived from the t-norm* T_0 *and* N' *is given by Equation (8).*

Proof. Let us suppose that I_{U_0} and N satisfy the U-Modus Tollens with respect to U. Then the first two items are a direct consequence of Proposition 4.2, whereas for the third one the same proof as in Theorem 3.1 works, taking into account that in our case, I_{T_0} just coincides with I' given in Equation (7).

Conversely, suppose that conditions 1-3 are satisfied. We will prove the U-Modus Tollens in some steps.

i) If $y \geq e$, we have $N(y) = 0$ and so the U-Modus Tollens holds.

ii) Let us suppose that $y < e$ and $x \leq y$. In this case $I_{U_0}(x, y) = 1$, and

$$U(N(y), I_{U_0}(x, y)) = U(N(y), 1) = N(y) \leq N(x).$$

iii) Let us suppose that $y < e$ and $y < x$. We distinguish two subcases:

– When $x > e$, we get

$$U(N(y), I_{U_0}(x, y)) = U(N(y), y) = 0 \leq N(x).$$

– When $x \leq e$, we have

$$U(N(y), I_{U_0}(x, y)) = U\left(eN'\left(\frac{y}{e}\right), eI_{T_0}\left(\frac{x}{e}, \frac{y}{e}\right)\right) = eT_U\left(N'\left(\frac{y}{e}\right), I_{T_0}\left(\frac{x}{e}, \frac{y}{e}\right)\right).$$

Now, if we apply Property 3, we obtain

$$U(N(y), I_{U_0}(x, y)) \leq eN'\left(\frac{x}{e}\right) = N(x). \quad \square$$

Remark 4.1. Note that Condition 1 in the previous theorem is quite natural. For instance, when the underlying t-norm T_U is continuous, the first part of Condition 1 just says that T_U must be a non-strict Archimedean t-norm with associated negation $N_{T_U} \geq N'$. Note that the second part of Condition 1 is always guaranteed when U is a uninorm in $\mathcal{U}_{\min}$. Moreover, if N takes all values in $]0, e[$, then Condition 1 holds only if U is a uninorm in $\mathcal{U}_{\min}$.

On the other hand, note also that again the underlying t-conorms S_0 and S_U play no role in the satisfaction of the U-Modus Tollens.

In the other two cases, $e < e_0$ and $e_0 < e$, we need to suppose that some of the underlying operators are given by an ordinal sum, as follows.

Theorem 4.2. *Let I_{U_0} be the RU-implication derived from $U_0 \equiv \langle T_0, e_0, S_0\rangle_{\min}$, N a fuzzy negation and $U \equiv \langle T_U, e, S_U\rangle$ a conjunctive uninorm with neutral element e with $e < e_0$. Suppose that U_0 is given by*

$$U_0(x, y) = \begin{cases} eT_0'\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } x, y \in [0, e], \\ e + (e_0 - e)T_0''\left(\frac{x-e}{e_0-e}, \frac{y-e}{e_0-e}\right) & \text{if } x, y \in [e, e_0], \\ e_0 + (1 - e_0)S_0\left(\frac{x-e_0}{1-e_0}, \frac{y-e_0}{1-e_0}\right) & \text{if } x, y \in [e_0, 1], \\ \min(x, y) & \text{otherwise}, \end{cases} \tag{11}$$

where T_0', T_0'' are t-norms. Then I_{U_0} and N satisfy the U-Modus Tollens with respect to U if and only if the following properties hold

1. *$U(N(y), y) = 0$ for all $y \leq e$ and $U(N(y), 1) = N(y)$ for all $y < e$.*
2. *$N(x) < e$ for all $x > 0$ and $N(x) = 0$ for all $x \geq e$.*
3. *$I_{T_0'}$ and N' satisfy the Modus Tollens with respect to the t-norm T_U for all $y < x$, where $I_{T_0'}$ is the residual implication derived from the t-norm T_0' and N' is given by Equation (8).*

Proof. According to Equation (11) we have $U_0(e, e) = e$ and for all $y < x \leq e$ we get

$$I_{U_0}(x, y) = eI_{T_0'}\left(\frac{x}{e}, \frac{y}{e}\right)$$

and the result follows similarly to the proof of Theorem 4.1. $\square$

Remark 4.2. Note that the structure of U_0 required in Equation (11) is not restrictive at all. In fact, since U_0 is in $\mathcal{U}_{\min}$, when T_0 is continuous then U_0 is given by Equation (11) if and only if e is an idempotent element of U_0, that is, $U_0(e, e) = e$ (and the corresponding t-norms T_0' and T_0'' are also continuous).

Theorem 4.3. *Let I_{U_0} be the RU-implication derived from $U_0 \equiv \langle T_0, e_0, S_0\rangle_{\min}$, N a fuzzy negation, and $U \equiv \langle T_U, e, S_U\rangle$ a conjunctive uninorm with neutral element e with $e_0 < e$. Suppose that the restriction of U to the square $[0, e]^2$ is given by*

$$U(x,y) = \begin{cases} e_0 T'_U\left(\frac{x}{e_0}, \frac{y}{e_0}\right) & \text{if } x, y \in [0, e_0], \\ e_0 + (e - e_0)T''_U\left(\frac{x-e_0}{e-e_0}, \frac{y-e_0}{e-e_0}\right) & \text{if } x, y \in [e_0, e], \\ \min(x, y) & \text{otherwise}, \end{cases} \tag{12}$$

where T'_U, T''_U are t-norms. Then I_{U_0} and N satisfy the U-Modus Tollens with respect to U if and only if the following properties hold

1. *$U(N(y), y) = 0$ for all $y \le e_0$ and $U(N(y), 1) = N(y)$ for all $y < e_0$.*
2. *$N(x) < e_0$ for all $x > 0$ and $N(x) = 0$ for all $x \ge e_0$.*
3. *I_{T_0} and N' satisfy the Modus Tollens with respect to the t-norm T'_U for all $y < x$, where I_{T_0} is the residual implication derived from the t-norm T_0 and N' is given by*

$$N'(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{N(e_0 x)}{e_0} & \text{otherwise}. \end{cases} \tag{13}$$

Proof. In this case, Item 2 is guaranteed by Item 5 in Proposition 4.2, whereas the rest of the proof is similar to the theorems above. □

Remark 4.3. Note that the requirement that the restriction of U to the square $[0, e]^2$ is given by Equation (12) is equivalent to be e_0 an idempotent element of U whenever the corresponding underlying t-norm T_U is continuous.

Example 4.1. Let us consider the uninorm in $\mathcal{U}_{\min}$ given by $U_0 \equiv \langle T_0, e_0, S_0\rangle_{\min}$ and the uninorm U defined by Theorem 4.3, where $T_0 = T'_U = T_{\mathbf{LK}}$ is the Łukasiewicz t-norm and S_0, S_U are any t-conorms. Let N be the negation given by

$$N(x) = \begin{cases} 1 & \text{if } x = 0, \\ e_0 - x & \text{if } 0 < x < e_0, \\ 0 & \text{if } x \ge e. \end{cases}$$

It is easy to see that these operators satisfy all the conditions of Theorem 4.3 and therefore I_{U_0}, N satisfy the U-Modus Tollens with respect to U.

4.2. Case of RU-implications derived from representable uninorms

In this section, we will consider the case of RU-implication obtained from representable uninorms, that is, from uninorms $U_0 \equiv \langle h, e_0\rangle_{\text{rep}}$ whose expression is given in Definition 2.5. Let us recall in this case the expression of the residual implication derived from U_0.

Proposition 4.3. (Theorem 5.4.10 in [3]) *Let $U_0 \equiv \langle h, e_0\rangle_{rep}$ be a representable uninorm with neutral element $e_0 \in]0, 1[$. Then I_{U_0} is given by*

$$I_{U_0}(x,y) = \begin{cases} 1 & \text{if } (x, y) \in \{(0, 0), (1, 1)\}, \\ h^{-1}(h(y) - h(x)) & \text{otherwise}. \end{cases} \tag{14}$$

In this section we will see that, contrariously to what happens in all the previous cases, in this one there are solutions of the U-Modus Tollens involving continuous (even strong) negations. In fact, we will characterize all solutions in the case when the negation is the proper strong negation associated to the representable uninorm. That is, given the uninorm $U_0 \equiv \langle h, e_0\rangle_{\text{rep}}$, the negation N is given by

$$N(x) = N_h(x) = h^{-1}(-h(x)) \qquad \text{for all} \quad x \in [0, 1].$$

Theorem 4.4. *Let $U_0 \equiv \langle h, e_0\rangle_{rep}$ be a representable uninorm, I_{U_0} its residual implication, and N_h its associated negation. Let U be a left-continuous conjunctive uninorm with neutral element e and I_U its residual implication. Then, the following conditions are equivalent:*

1) *I_{U_0} and N_h satisfy the U-Modus Tollens with respect to U.*
2) *$I_{U_0}(x, y) \le I_U(N_h(y), N_h(x))$ for all $x, y \in [0, 1]$.*
3) *$U(x, y) \le U_0(x, y)$ for all $x, y \in [0, 1]$.*

Proof. Equivalence between (1) and (2) follows directly from the general result given in Proposition 3.2. So, we only need to prove the equivalence between (2) and (3). However, it is well known (see Proposition 5.4.13 in [3]) that I_{U_0} satisfies contrapositive symmetry with respect to N_h, that is,

$$I_{U_0}(x,y) = I_{U_0}(N_h(y), N_h(x)) \quad \text{for all } x, y \in [0,1].$$

Thus, taking into account that N_h is a strong negation, and therefore a bijective function, we get that (2) is equivalent to

$$I_{U_0}(x,y) \le I_U(x,y) \quad \text{for all } x, y \in [0,1].$$

Finally, the equivalence between this last inequality and (3) follows from the fact that, for RU-implications derived from left-continuous uninorms, it holds

$$I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \le y\} \quad \text{and} \quad U(x,y) = \inf\{z \in [0,1] \mid I_U(x,z) \ge y\}. \quad \square$$

Example 4.2. Let $U_0 \equiv \langle h, e\rangle_{\text{rep}}$ be a representable uninorm with neutral element $e \in]0,1[$. We know that the underlying t-norm T_{U_0} and t-conorm S_{U_0} are strict. Let us consider $U, U' \in \mathcal{U}_{\min}$ given by

$$U \equiv \langle T_{U_0}, e, S_{U_0}\rangle_{\min} \qquad \text{and} \qquad U' \equiv \langle \min, e, S_{U_0}\rangle_{\min}.$$

It is obvious that $U \le U_0$ but $U' \not\le U_0$ and then, according to Theorem 4.4, we obtain that I_{U_0}, N_h satisfy U-Modus Tollens with respect to U but not with respect to U'.

4.3. *Case of RU-implications derived from idempotent uninorms*

In this section, we will deal with RU-implications derived from idempotents uninorms, that is, from uninorms $U_0 \equiv \langle g, e_0\rangle_{\text{ide}}$ with neutral element $e_0 \in [0,1]$ and such that $g(0) = 1$. We will see that in this case solutions involving continuous negations are again available. Let us recall in this case the expression of the residual implication derived from U_0.

Proposition 4.4. (Theorem 5.4.14 in [3]) *Let $U_0 \equiv \langle g, e_0\rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e_0 \in]0,1[$ and such that $g(0) = 1$. Then I_{U_0} is given by*

$$I_{U_0}(x,y) = \begin{cases} \max(g(x), y) & \text{if } x \le y, \\ \min(g(x), y) & \text{if } x > y. \end{cases} \tag{15}$$

In the particular case of left-continuous idempotent uninorms with associated function g given by a strong negation, it is possible to obtain a similar result to the one obtained in the case of representable uninorms (see Theorem 4.4) with a very similar proof.

Theorem 4.5. *Let N_0 be a strong fuzzy negation with fixed point e_0. Let $U_0 \equiv \langle N_0, e_0\rangle_{\text{ide}}$ be a left-continuous idempotent uninorm with neutral element $e_0 \in]0,1[$ and I_{U_0} its residual fuzzy implication. Let U be a left-continuous conjunctive uninorm with neutral element e and I_U its residual implication. Then the following conditions are equivalent:*

1) *I_{U_0} satisfies U-Modus Tollens with respect to U and N_0.*
2) *$I_{U_0}(x,y) \le I_U(N_0(y), N_0(x))$ for all $x, y \in [0,1]$.*
3) *$U(x,y) \le U_0(x,y)$ for all $x, y \in [0,1]$.*

In the general case when g is simply a decreasing function symmetric with respect to the main diagonal, we will distinguish two cases depending on whether $g(1) = 0$ or not. First, we deal with the case $g(1) = 0$ and we give only a partial result when both, the neutral element of U and the neutral element of U_0, coincide with the fixed point of the negation N.

Theorem 4.6. *Let $U_0 \equiv \langle g, e_0\rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e_0 \in]0,1[$ such that $g(0) = 1$ and $g(1) = 0$. Let N be a fuzzy negation with fixed point $e_N = e_0$ and let U be a uninorm with neutral element $e = e_0$. Then I_{U_0} and N satisfy U-Modus Tollens with respect to U if and only if the following properties hold:*

1. *$g(x) \le N(x)$ for all $x \in [0,1]$.*
2. *$U(N(x), g(x)) = N(x)$ for all $x \le e_0$.*
3. *$U(N(x), x) = N(x)$ for all $x \ge e_0$.*
4. *$U(N(y), y) \le N(x)$ for all $y \le g(x) \le e_0 \le x$.*

Proof. Let us suppose first that I_{U_0} and N satisfy U-Modus Tollens with respect to U and let us prove conditions 1-4 step by step.

1. If we take $y = e_0$ in Equation (6) we obtain
$$U(N(e_0), I_{U_0}(x, e_0)) = U(e_0, g(x)) = g(x) \le N(x)$$
for all $x \in [0, 1]$.
2. Putting in this case $x = y$ in Equation (6) and taking into account that $x \le e_0$, we have
$$U(N(x), I_{U_0}(x, x)) = U(N(x), g(x)) \le N(x).$$
However, since $x \le e_0$, we have that both $N(x), g(x) \ge e_0$ and then
$$U(N(x), g(x)) \ge \max(N(x), g(x)) = N(x),$$
and hence we obtain the equality.
3. Similarly as above, taking $x = y$ in Equation (6) we obtain in this case
$$U(N(x), I_{U_0}(x, x)) = U(N(x), x) \le N(x).$$
Now, since $x \ge e_0$ we have $U(N(x), x) \ge U(N(x), e_0) = N(x)$ and we derive again the equality.
4. Finally, suppose that $y \le g(x) \le e_0 \le x$. Then it is $y \le g(x) \le e_0 \le g(y)$ and then we obtain by the U-Modus Tollens that
$$U(N(y), I_{U_0}(x, y)) = U(N(y), y) \le N(x).$$

Conversely, suppose that conditions 1-4 hold and let us prove the U-Modus Tollens with respect to U by distinguishing some cases:

(i) When $x \le y \le g(x)$. In this case we have $N(y) \le N(x)$ and $x \le e_0$ and then
$$U(N(y), I_{U_0}(x, y)) = U(N(y), g(x)) \le U(N(x), g(x)) = N(x),$$
where last equality follows by Condition 2.

(ii) When $x, g(x) \le y$. Similarly, in this case $N(y) \le N(x)$, $y \ge e_0$ and then
$$U(N(y), I_{U_0}(x, y)) = U(N(y), y) = N(y) \le N(x),$$
where the last equality is due to Condition 3.

(iii) When $g(x) \le y < x$. In this case it must be $e_0 \le x$ and also $g(x) \le N(x) \le e_0$ and $N(y) \le N(g(x))$. Consequently, we obtain
$$U(N(y), I_{U_0}(x, y)) = U(N(y), g(x)) \le U(N(g(x)), g(x)) \le N(x),$$
where the last inequality follows by Condition 4.

(iv) When $x, g(x) > y$. In this last case we have $y < e_0 \le N(y)$ and then we can consider two possibilities:
- If $x \le e_0$ then $N(x) \ge e_0$ and considering Condition 4 with $x = e_0$ we derive $U(N(y), y) \le e_0$, from which we deduce
$$U(N(y), I_{U_0}(x, y)) = U(N(y), y) \le e_0 \le N(x).$$
- If $x > e_0$ then $N(x) \le e_0$ and again from Condition 4 we have
$$U(N(y), I_{U_0}(x, y)) = U(N(y), y) \le N(x). \quad \square$$

Note that there are a lot of uninorms U and negations N satisfying all conditions given in the previous theorem. As a mater of example let us describe two particular cases in the following theorem.

Theorem 4.7. *Let $U_0 \equiv \langle g, e_0\rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e_0 \in]0, 1[$ such that $g(0) = 1$ and $g(1) = 0$. Let N be a fuzzy negation with fixed point $e_N = e_0$ and let U be a uninorm with neutral element $e = e_0$. Then the following properties hold:*

1. *If $U \equiv \langle T_U, e_0, S_U\rangle_{\min}$ is a uninorm in $\mathcal{U}_{\min}$ with underlying t-conorm $S_U = \max$, then I_{U_0} and N satisfy U-Modus Tollens with respect to U if and only if $g(x) \le N(x)$ for all $x \in [0, 1]$.*
2. *If N is strong and $U \equiv \langle N, e_0\rangle_{\text{ide}}$ is the corresponding left-continuous idempotent uninorm with associated function N, then I_{U_0} and N satisfy U-Modus Tollens with respect to U if and only if $g(x) \le N(x)$ for all $x \in [0, 1]$.*

Proof. We only need to prove that in both cases the remaining Conditions 2-4 in Theorem 4.6 always hold.

1. If $U \equiv \langle T_U, e_0, S_U \rangle_{\min}$ Conditions 3-4 follow directly, whereas Condition 2 is also guaranteed when $S_U = \max$.
2. If N is strong and $U \equiv \langle N, e_0 \rangle_{\text{ide}}$ is left-continuous it holds $U(N(x), x) = \min(N(x), x)$ ensuring conditions 3-4. On the other hand, when $x \le e_0$ we have $g(x), N(x) \ge e_0$ and then

   $$U(N(x), g(x)) = \max(N(x), g(x)) = N(x),$$

   ensuring Condition 2. □

Let us deal now with the case of RU-implications built from idempotent uninorms $U_0 \equiv \langle g, e_0 \rangle_{\text{ide}}$ with neutral element $e_0 \in]0, 1[$ and such that $g(0) = 1$ and $g(1) > 0$. We will split our reasoning into two cases depending on whether $g(1) \ge e$ or not. Note that in any case it must be $g(1) \le e_0 = g(e_0)$ because g is decreasing.

Theorem 4.8. *Let $U_0 \equiv \langle g, e_0 \rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e_0 \in]0, 1[$ such that $g(0) = 1$ and $g(1) = \alpha > 0$. Let N be a fuzzy negation and U a uninorm with neutral element e with $e \le \alpha \le e_0$. Then I_{U_0} and N satisfy the U-Modus Tollens with respect to U if and only if the following properties hold:*

1. *$U(N(y), y) = 0$ for all $y \le e$.*
2. *$N(x) < e$ for all $x > 0$ and $N(x) = 0$ for all $x \ge e$.*
3. *$U(N(x), 1) = N(x)$ for all $x \le e$.*

Proof. Let us suppose first that I_{U_0} and N satisfy U-Modus Tollens with respect to U.

1. Since $I_{U_0}(1, y) = y$ for all $y \le g(1) = \alpha$ and $e \le \alpha$, taking $x = 1$ in Equation (6) we obtain

   $$U(N(y), I_{U_0}(1, y)) = U(N(y), y) \le N(1) = 0$$

   and then $U(N(y), y) = 0$ for all $y \le e$.
2. Taking $x = e \le \alpha$ in the previous step, we have $U(N(e), e) = N(e) = 0$ and so $N(x) = 0$ for all $x \ge e$ because N is a decreasing function. Moreover, the fact that $N(x) < e$ for all $x > 0$ follows directly from Proposition 3.3 because $I_{U_0}(1, y) \ne 0$ for all $y > 0$.
3. Since $\alpha \le e_0$, taking $x = y < \alpha$ in Equation (6) (by symmetry with respect to the main diagonal it must be $g(x) = 1$ for all $x < \alpha$. Thus, $I_{U_0} = (x, x) = \max(g(x), x) = 1$), we obtain:

   $$U(N(x), I_{U_0}(x, x)) = U(N(x), 1) \le N(x).$$

   Taking into account that $U(N(x), 1) \ge N(x)$ holds for any uninorm U, we derive the equality for all $x \le e \le \alpha$.

Conversely, suppose that conditions 1-3 hold and we will split the proof in some cases.

(i) When $y \ge e$ we have $N(y) = 0$ and the U-Modus Tollens follows because U is conjunctive.
(ii) When $y < e \le \alpha$, in this case, we distinguish two cases:
  - If $x \le y$ then $U(N(y), I_{U_0}(x, y)) = U(N(y), 1) = N(y) \le N(x)$.
  - If $y < x$ then $U(N(y), I_{U_0}(x, y)) = U(N(y), y) = 0 \le N(x)$. □

Remark 4.4. Again there are a lot of uninorms U and negations N satisfying all conditions in the theorem above as we can see in next example. Just note that,

- Condition 1 only affects the underlying t-norm T_U. In particular, the zero region of T_U must contain all points $(N(y), y)$ with $y \le e$. For instance, when T_U is continuous this implies that T_U must be a non-strict Archimedean t-norm with associated negation N_{T_U} such that $N_{T_U}(y) \ge \frac{N(ey)}{e}$ for all $y \in [0, 1]$.
- Condition 3 in Theorem 4.8 holds for instance whenever U is a uninorm in $\mathcal{U}_{\min}$.

Example 4.3. Let $U_0 \equiv \langle g, e_0 \rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e_0 \in]0, 1[$ such that $g(0) = 1$ and $g(1) = \alpha > 0$. Take some $e \le \alpha$ and consider the uninorm $U = \langle T_{\mathbf{LK}}, e, S_U \rangle_{\min}$ and the negation N_e defined in Example 3.2. According to the previous remark, U and N_e verify all conditions in Theorem 4.8 and so, I_{U_0} and N_e satisfy the U-Modus Tollens with respect to U.

Finally, we deal with the case $0 < g(1) = \alpha < e$. In this framework we will reduce our study to the case when $e = e_0$. Even with this restriction, the characterization in that case becomes subjected to five conditions as we prove in the following result.

Theorem 4.9. *Let $U_0 \equiv \langle g, e_0\rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e_0 \in]0,1[$ such that $g(0) = 1$ and $g(1) = \alpha > 0$. Let N be a fuzzy negation and U a uninorm with neutral element $e = e_0$. Then I_{U_0} and N satisfy the U-Modus Tollens with respect to U if and only if the following properties hold:*

1. $U(N(y), y) = 0$ *for all* $y \le \alpha$.
2. $N(x) < e_0$ *for all* $x > 0$.
3. $U(N(x), g(x)) = N(x)$ *for all* $x \le e_0$.
4. $U(N(y), g(x)) \le N(x)$ *for all* $\alpha \le g(x) \le y < x$.
5. $U(N(y), y) \le N(x)$ *for all* $x, g(x) \le y$, *and also for all* $y < x$ *and* $\alpha \le y \le g(x) \le e_0 \le x$.

Proof. Let us suppose that I_{U_0} and N satisfy the U-Modus Tollens with respect to U and we prove the items one by one.

1. This item follows similarly as in the previous theorem just taking $x = 1$ in Equation (6).
2. This item follows directly from Proposition 3.3 because $I_{U_0}(1, y) \ne 0$ for all $y > 0$.
3. Note first of all that taking $x = y$ in Equation (6) we have

$$U(N(x), I_{U_0}(x,x)) = U(N(x), g(x)) \le N(x) \quad \text{for all } x \in [0,1]. \tag{16}$$

However, if we consider $x \le e_o$ we have $g(x) \ge e_0$ and then it is also

$$U(N(x), g(x)) \ge U(N(x), e_0) = N(x),$$

and hence the item follows.

4. If $\alpha \le g(x) \le y < x$ we have again by Equation (6) that

$$U(N(y), I_{U_0}(x,y)) = U(N(y), g(x)) \le N(x).$$

5. If $x, g(x) \le y$ by Equation (6) we obtain

$$U(N(y), I_{U_0}(x,y)) = U(N(y), y) \le N(x).$$

If $y < x$ and $\alpha \le y \le g(x) \le e \le x$ then $U(N(y), I_{U_0}(x,y)) = U(N(y), y) \le N(x)$.

Conversely, suppose that properties 1-5 are fulfilled and we will prove that I_{U_0} and N satisfy U-Modus Tollens with respect to U by considering some cases.

(i) When $x \le y \le g(x)$, we have $N(y) \le N(x)$ and $x \le e_0$. Then

$$U(N(y), I_{U_0}(x,y)) = U(N(y), g(x)) \le U(N(x), g(x)) = N(x)$$

where last equality follows by Property 3.

(ii) When $x, g(x) \le y$, applying Property 5 we have

$$U(N(y), I_{U_o}(x,y)) = U(N(y), y) \le N(x).$$

(iii) When $\alpha \le g(x) \le y < x$, we have

$$U(N(y), I_{U_0}(x,y)) = U(N(y), g(x)) \le N(x)$$

according to Property 4.

(iv) When $x, g(x) > y$, we have $U(N(y), I_{U_0}(x,y)) = U(N(y), y)$ and then we can consider two cases:
- If $y \le \alpha$ then by Property 1 we have $U(N(y), y) = 0 \le N(x)$.
- If $y > \alpha$ then $U(N(y), y) \le N(x)$ according to Property 5. □

5. Conclusions and future work

Modus Tollens is an inference rule which becomes essential when backward reasoning is applied. When Modus Tollens is want to be applied in fuzzy control and approximate reasoning, the fuzzy logical operators used in this framework must satisfy the inequality

$$T(N(y), I(x,y)) \le N(x) \quad \text{for all} \quad x, y \in [0,1],$$

where T is a t-norm, I is an implication function and N is a negation. Due to its importance, the previous equation has been extensively investigated and even some generalizations have appeared.

In this paper we have generalized this inequality by substituting the t-norm T by a conjunctive uninorm U, leading to the U-Modus Tollens. This new property has been exhaustively studied in many situations and, specially, in the case when I satisfies (NP), which is the case of the most usual implications. Moreover, a special attention has been paid to the case of RU-implications, that is, residuated implications derived from uninorms. A lot of solutions have been characterized when RU-implications are derived from uninorms in $\mathcal{U}_{\min}$, from representable uninorms, and from idempotent uninorms, including many examples.

As a future work, this study can be extended to RU-implications derived from other kinds of uninorms, like compensatory uninorm or uninorms continuous in the open unit square. Also other types of implications can be considered as (U, N)-implications (see [4]), or h-implications and (h, e)-implications (see [21]).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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