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A review on logical connectives defined on finite chains

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Abstract

This paper aims to compile the main results published in the literature on the study of logical connectives defined on finite chains, as well as to establish relationships between different subfamilies and to set up a common notation for all of them. The operators defined on finite chains which are analyzed are conjunctions and disjunctions such as t-norms, t-conorms, copulas, uninorms, among others; also, results related to implications defined also on finite chains are additionally gathered. For all the theoretical properties, it is mentioned which have been completely or partially solved or, on the contrary, which remain as open problems.

Keywords: Finite chains; Logical connectives; Implications on finite chains; t-norms on finite chains; Uninorms on finite chains

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1. Introduction

Connectives on finite domains have been extensively developed in last decades from the points of view of theory and applications. From the theoretical point of view, these connectives allow to handle qualitative information directly, avoiding the numerical conversion between the finite scale and the real unit interval, which has traditionally been the main domain of connectives in fuzzy logic. Among the connectives on finite domains, this review focuses on those that are defined on finite totally ordered scales with $n+1$ elements, represented with the generic finite chain $L_n = \{0, 1, \ldots, n\}$. Although these operators have been also called discrete operators by some authors, the term may be misleading since discrete domains are not necessarily finite nor totally ordered. For this reason, in this paper we will refer to these operators as operators on L_n. Furthermore, for the sake of readability, when no ambiguity may appear, we will omit explicitly the domain L_n, taking into account that the whole paper is devoted to these operators. We highlight mainly two families of operators which will be studied: conjunctions and disjunctions, and implications. On the one hand, conjunctions and disjunctions encompass important families of operators such as t-norms, t-conorms,

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copulas and uninorms (which generalize t-norms and t-conorms). Furthermore, they can be understood as multivalued functions whose output summarizes, or represents, all the input information. On the other hand, implications are used to generalize the conditional statement *if p then q*. From the point of view of applications, connectives on L_n are the suitable operators for processing information that can be stored on a computer, such as images. One of the main applications in image processing is discrete mathematical morphology, which uses operators having L_n as domain that depend on conjunctions, disjunctions and implications. Moreover, by the nature of conjunctions and disjunctions, they allow to study and draw conclusions about imprecise qualitative information, such as the final evaluation of experts about a certain system.

Different surveys dealing with connectives on L_n can be found in the literature. In particular, in [1] there is a survey about t-norms and copulas, from which we collect the main results on these operators; however, the relations between both families of operators are not exposed in depth. In [2] there is a survey about uninorms in general, where the main subfamilies of uninorms on L_n are presented and related to their counterpart in $[0,1]$. Nevertheless, the connections between the different families of uninorms on L_n are not presented. Finally, in [3,4] there are two surveys about implication functions on $[0,1]$ and on finite chains. In both, the vast majority of the results involve implications on $[0,1]$. For these reasons, this review draws attention on the most important logical connectives defined on finite chains. The aim is twofold. On the one hand, to bring together in a single paper the most relevant results on different families of connectives with a common notation for all of them. On the other hand, to complement the existing surveys in different ways: for t-norms, to show some construction techniques, both from the point of view of the restriction of t-norms in $[0,1]$ to values on a finite chain, and from the point of view of additive generators; for copulas, to present the relationships between these and t-norms, since there are axioms in common in the definitions of both operators; for uninorms, because a large number of subclasses have been introduced in the literature, a classification of all of them has been carried out, showing which ones are subsets of others and studying the intersections of some of them; finally, for implications, the conditions that some families of these operators must satisfy in order to fulfill some additional properties of implications have been compiled and summarized.

With this, the paper is structured as follows. Section 2 sets out the common notation to be used throughout the review, as well as the definitions of basic connectives. Section 3 studies t-norms and t-conorms and the duality between them, allowing to study and transfer properties from one family to the other. After that, in Section 4 uninorms are presented as the operators that generalize t-norms and t-conorms, and the different subfamilies are also presented. Section 5 continues with another family of aggregation functions: copulas. Due to their definition, they also share many properties with t-norms. Finally, with respect to conjunctions and disjunctions, some related operators are defined in Section 6. Finally, leaving aside these operators, Section 7 sets out the properties of implications and the different existing families.

2. Negations, conjunctions and disjunctions

On this paper we study different operators defined on finite chains. Given an ordered finite set with $n+1$ elements, there always exists a bijection between itself and the finite chain $L_n = \{0, 1, \ldots, n-1, n\}$, equipped with the usual order. Therefore, it is enough to set as the scope of the study those operators defined over the set L_n with $n \geqslant 1$. Furthermore, we denote by $[x, y]$ the finite subchain of L_n whose elements are $z \in L_n$ such that $x \leqslant z \leqslant y$.

Among all operators defined on L_n, in this paper we make a deeper study on negations, implications, conjunctions and disjunctions. In this section we will establish the basics in order to go into detail about each family in its corresponding section.

The first concept we present is the one of negation, which generalizes the usual logical negation for finite chains.

Definition 2.1. ([5]) A *negation on L_n* is an unary operator $N : L_n \to L_n$ such that satisfies the following conditions:

(N1) $N(x) \geqslant N(y)$ for all $x \leqslant y$,
(N2) $N(0) = n$ and $N(n) = 0$.

Furthermore, a negation is *strong* if it is involutive; that is, $N(N(x)) = x$ for all $x \in L_n$.

Example 2.1. The least negation, N_{D_1}, and the greatest negation, N_{D_2}, are negations that are not strong:

$$N_{D_1}(x) = \begin{cases} n, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases} \qquad N_{D_2}(x) = \begin{cases} n, & \text{if } x \neq n, \\ 0, & \text{if } x = n. \end{cases}$$

Following Definition 2.1, it can be deduced that any involution is a bijective operator. Therefore, there exists a unique operator that is a strong negation on L_n (see Proposition 2.1). Note that this property is not true in the [0, 1] framework since there exist infinite strong negations [6].

Proposition 2.1. *([5]) The only strong negation on L_n is the* classical negation *given by*

$$N_C(x) = n - x, \qquad \textit{for all } x \in L_n. \tag{1}$$

Attending to this property, the classical negation given in Equation (1) acts as the only automorphism in L_n with the negation properties, and therefore will allow us to establish dualities between families of operators. In addition, there is the possibility of generalizing the concept of negation, finding in [7] a more flexible definition.

The second concept we introduce is the one of conjunction and disjunction, which can be understood as particular cases of aggregation functions. Aggregation functions are a family of operators that are widely studied because they are used in numerous practical cases, from aggregation of information [8] to image processing [9].

Definition 2.2. ([1]) A mapping $F : L_n^q \to L_n$ is said to be an *aggregation function on L_n* if it is increasing in each argument and satisfies the boundary conditions $F(0, \dots, 0) = 0$ and $F(n, \dots, n) = n$.

Definition 2.2 gives us the concept of q-dimensional aggregation function, since the main objective of these operators is to combine several inputs. However, the vast majority of operators in the paper are formulated for the case $q = 2$, which we will call binary aggregation functions or, simply, aggregation functions.

Due to the simplicity of its definition and the wide range of situations in which it is used, the scientific community has devoted its efforts to study different interesting properties and their possible use in practical cases. In the [0, 1] case, a well-studied property of aggregation functions is continuity. Naturally, this property is not available when L_n is the domain of the operator. Therefore, an attempt to translate the concept of continuity to the L_n domain is made through the concept of smoothness. The idea of smoothness appears for the first time in [10] by studying linguistic labels in decision making under the idea that subtle changes in the labels should derive into subtle changes in the final assessment.

Definition 2.3. ([5,11]) Let $f : L_n \to L_n$ be a unary operator. It is said that f is *k-smooth* (or *smooth* simply when $k = 1$) if

$$0 \leqslant |f(x+1) - f(x)| \leqslant k, \qquad \text{for all } x \in L_n \setminus \{n\}. \tag{2}$$

According to this definition of smoothness, the idea of continuity in the L_n domain corresponds to the 1-smooth case.

Remark 2.1. Before continuing, we would like to discuss the linguistic meaning that Equation (2) can take because these operations will be frequent throughout the review. At the beginning of the section it has been established that there is always a bijection between an ordered finite set S with $n + 1$ elements and the finite chain L_n. Note, however, that the set S may not be uniformly distributed, which is the case for L_n. In this sense, by performing the operation $f(x + 1)$, we are actually claiming from the point of view of the structure of S that, given an element at position x, represented by s_x, we are evaluating the transformation f at the next value of the set, represented by s_{x+1}. Also, the value k is intended to set the maximum number of steps or linguistic labels between the value of $f(s_{x+1})$ and $f(s_x)$.

Having made this comment so that the reader is aware, we continue introducing some properties that an aggregation function can satisfy, which are listed in the following definition:

Definition 2.4. ([1,5,12,13]) Let $F : L_n \times L_n \to L_n$ be an aggregation function. We say that F

(i) is *divisible* if for all $x, y \in L_n$ with $x \leqslant y$, there are $z, z' \in L_n$ such that $x = F(y, z) = F(z', y)$.
(ii) is k-*smooth* (or *smooth* simply when $k = 1$) if it is k-smooth in each variable; that is,

$$F(x+1, y) - F(x, y) \leqslant k, \quad \text{for all } x \in L_n \setminus \{n\} \text{ and } y \in L_n,$$
$$F(x, y+1) - F(x, y) \leqslant k, \quad \text{for all } y \in L_n \setminus \{n\} \text{ and } x \in L_n.$$

(iii) is a *kernel* if $F(x, y) - F(x', y') \leqslant \max\{x - x', y - y'\}$ for all $x, y, x', y' \in L_n$ such that $x' \leqslant x$ and $y' \leqslant y$.
(iv) is *idempotent-free* if the only idempotent elements[1] of F are 0 and n.
(v) is *conservative* if $F(x, y) \in \{x, y\}$ for all $x, y \in L_n$.

When the aggregation function is commutative, representing that it does not matter in which order the inputs are combined, additional properties can be obtained and are collected in the following definition:

Definition 2.5. ([1,5]) Let $F : L_n \times L_n \to L_n$ be a commutative aggregation function. We say that F

(i) is *Archimedean* if it satisfies the following condition: For all $x, y \in L_n \setminus \{0, n\}$ there is $m \in \mathbb{N}$ such that $x_F^{(m)} < y$, where $x_F^{(m)}$ is defined recursively as

$$x_F^{(m)} = \begin{cases} x, & \text{if } m = 1, \\ F(x_F^{(m-1)}, x), & \text{otherwise.} \end{cases} \tag{3}$$

(ii) satisfies the 1-*Lipschitz condition* if for all $x, y, z \in L_n$, $F(z, y) - F(x, y) \leqslant z - x$ for all $z \geqslant x$.
(iii) satisfies the *Intermediate Value Theorem*; that is, let $z = F(x, y)$ and let $z' = F(x, y')$ with $z < z'$, then for all $z'' \in [z, z']$ there exists some $y'' \in [y, y']$ such that $z'' = F(x, y'')$.

Of these three properties, the Archimedean is the least intuitive and the least verifiable. In the following proposition there is an alternative definition of the Archimedean property, which allows us to operate more easily.

Proposition 2.2. *([14]) Let $F : L_n \times L_n \to L_n$ be a commutative aggregation function. Then, F is Archimedean if, and only if, F satisfies that $F(x, x) < x$ for all $x \in L_n \setminus \{0, n\}$.*

Finally, as subfamilies of aggregation functions, we present conjunctions and disjunctions, which generalize the *and* and *or* operators of classical logic, i.e., they generalize the intersection and union of sets, respectively.

Definition 2.6. A binary operator $C : L_n \times L_n \to L_n$ is a *conjunction on L_n* if it is increasing in each argument and satisfies the following boundary conditions:

(C1) $C(0, n) = C(n, 0) = 0$,
(C2) $C(n, n) = n$.

Definition 2.7. A binary operator $D : L_n \times L_n \to L_n$ is a *disjunction on L_n* if it is increasing in each argument and satisfies the following boundary conditions:

(D1) $D(0, n) = D(n, 0) = n$,
(D2) $D(0, 0) = 0$.

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[1] An element $x \in L_n$ is said to be *idempotent* of an aggregation function F on L_n if $F(x, x) = x$.

3. Triangular norms and conorms

In this section we will study two particular cases of aggregation functions: triangular norms and triangular conorms, as two families of conjunctions and disjunctions, respectively.

Triangular norms are one of the most studied operators in fuzzy logic, both in the $[0,1]$ and the L_n cases. In addition, they have extensions in other fields such as multisets [15], product lattices [16] and discrete fuzzy numbers [17].

Definition 3.1. ([5]) A *triangular norm on* L_n (briefly *t-norm on* L_n) is a binary operator $T : L_n \times L_n \to L_n$ such that it is an associative and commutative conjunction with n as neutral element; that is, $T(x,n) = x$ for all $x \in L_n$.

From this definition, we highlight the importance of the associative property. When combining different inputs using binary aggregation functions, it is necessary to combine them by pairs. However, this technique has a drawback: the order in which the combination is performed is important, and obviously different aggregation orders can give different results. With the associativity, it is satisfied that $T(T(x,y),z) = T(x,T(y,z))$ for all $x,y,z \in L_n$; when combined with commutativity, it can be deduced that the way in which the initial inputs are aggregated is order-free. It is for this reason that t-norms (and other associative and commutative aggregation functions like t-conorms and uninorms) are of great importance in practical cases.

In the following example some t-norms on L_n are presented, which are the translation of their corresponding operator defined on $[0,1]$.

Example 3.1. Some examples of t-norms are:

- The minimum t-norm:

$$T_{\boldsymbol{M}}(x,y) = \min\{x,y\}.$$

- The drastic t-norm:

$$T_{\boldsymbol{D}}(x,y) = \begin{cases} 0, & \text{if } x,y \neq n, \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

- The nilpotent minimum t-norm:

$$T_{\boldsymbol{nM}}(x,y) = \begin{cases} 0, & \text{if } x+y \leqslant n, \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

- The Łukasiewicz t-norm:

$$T_{\boldsymbol{L}}(x,y) = \max\{0, x+y-n\}.$$

All these t-norms have their counterpart in the $[0,1]$ framework, although the reciprocal is not true. For example, the product t-norm $T_{\boldsymbol{P}}(x,y) = xy$, for all $x,y \in [0,1]$, is not well defined in the L_n case.

3.1. Smooth and Archimedean t-norms

In Definitions 2.4 and 2.5 some properties of aggregation functions have been defined. These properties have been studied for t-norms, and in the following proposition we have an equivalence between some of them:

Proposition 3.1. *([5]) Let T be a t-norm on L_n. The following statements are equivalent:*

- *T is divisible.*
- *T is smooth.*

- *T satisfies the 1-Lipschitz condition.*
- *T satisfies the Intermediate Value Theorem.*

Another property that is interesting to characterize is the Archimedean one. In Proposition 2.2 we have already given a first result that characterizes all commutative aggregation functions which are Archimedean. In the following proposition, we have a more specific result for t-norms:

Proposition 3.2. *([1,5]) Let T be a t-norm on L_n. The following statements are equivalent:*

- *T is Archimedean.*
- *T is idempotent-free; that is, $T(x, x) \neq x$ for all $x \in L_n \setminus \{0, n\}$,*
- *$T(x, y) \neq \min\{x, y\}$, for all $x, y \in L_n \setminus \{0, n\}$.*

It is important to remark that the equivalence presented in Proposition 3.2 only holds in the $[0, 1]$ case for continuous t-norms. From now on in this section, we denote by $\mathcal{T}_{\text{div}}$ the set of t-norms on L_n that satisfy the equivalent properties of Proposition 3.1. Analogously, we denote by $\mathcal{T}_{\text{arc}}$ the set of t-norms on L_n that satisfy the equivalent properties of Proposition 3.2.

A final question is to characterize the operators belonging to $\mathcal{T}_{\text{div}} \cap \mathcal{T}_{\text{arc}}$. In the following proposition it is shown that the intersection is non empty, but only exists a unique t-norm satisfying all these properties.

Proposition 3.3. *([5]) The only Archimedean smooth t-norm on L_n is the Łukasiewicz t-norm.*

Let us now study how to generate a t-norm from a given family of t-norms defined on a subset of L_n. Indeed, given some $r \in \{0, 1, \dots, n-1\}$, let $a_0, a_1, \dots, a_r, a_{r+1} \in L_n$ be a sequence of points such that $0 = a_0 < a_1 < \cdots < a_r < a_{r+1} = n$. Let T_i be a t-norm defined on each $[a_i, a_{i+1}]$, with $i \in J = \{0, 1, \dots, r\}$. Thus, from each pair $([a_i, a_{i+1}], T_i)$ we can consider the family $\{([a_i, a_{i+1}], T_i)\}_{i \in J}$ of t-norms defined over their corresponding interval, so we can construct a new t-norm on L_n using some results about ordinal sums of semigroups [18]. In the following definition we have the concept of ordinal sums of t-norms.

Definition 3.2. ([1,5]) Let $r \in L_n$, $J = \{0, 1, \dots, r\}$, and $\{([a_i, a_{i+1}], T_i)\}_{i \in J}$ be a family of t-norms, where each t-norm T_i is defined over the subinterval $[a_i, a_{i+1}]$ of L_n. Then, the operator T on L_n, defined by

$$T(x, y) = \begin{cases} T_i(x, y), & \text{if } (x, y) \in [a_i, a_{i+1}]^2 \text{ for some } i \in J, \\ \min\{x, y\}, & \text{otherwise}, \end{cases} \tag{4}$$

is called the *ordinal sum* of $\{([a_i, a_{i+1}], T_i)\}_{i \in J}$, and it is denoted by

$$T = \big\langle ([a_i, a_{i+1}], T_i)_{i \in J} \big\rangle.$$

Proposition 3.4. *([1,5]) Let $r \in L_n$, $J = \{0, 1, \dots, r\}$ and $\{([a_i, a_{i+1}], T_i)\}_{i \in J}$ be a family of t-norms, where each t-norm T_i is defined over the subinterval $[a_i, a_{i+1}]$ of L_n. Then, the ordinal sum $T = \big\langle ([a_i, a_{i+1}], T_i)_{i \in J} \big\rangle$ is a t-norm on L_n.*

The importance of studying t-norms on L_n defined as ordinal sums lies in the fact that we can analyze the properties that T has with respect to the properties that all T_i have and vice versa. Fig. 1 shows graphically the structure of the ordinal sums over L_n. Let us observe that the set containing 0, n and the end-points of each interval $[a_i, a_{i+1}]$, for all $i \in J$, is strictly contained in the set of idempotent elements of $T = \big\langle ([a_i, a_{i+1}], T_i)_{i \in J} \big\rangle$. The equality between these two sets occurs if, and only if, the set of idempotent elements of each t-norm on $[a_i, a_{i+1}]$ is $\{a_i, a_{i+1}\}$, and that occurs if, and only if, each T_i is idempotent-free and so Archimedean. Note that if $r > 0$, the t-norm T on L_n can never be Archimedean; moreover, T is Archimedean if, and only if, $r = 0$ and T_0 is Archimedean.

Another property to be studied for ordinal sums is the smoothness of T, which is characterized through its structure in the next proposition.

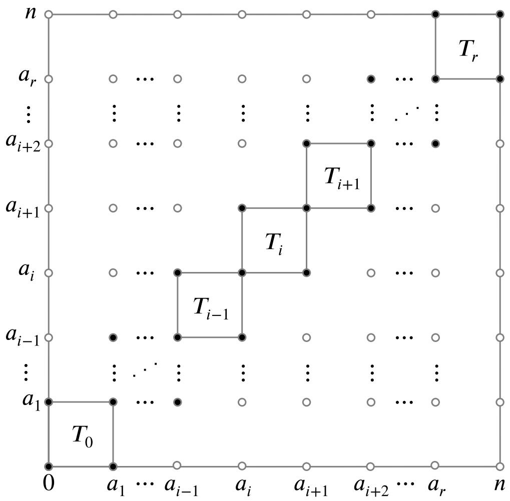

Fig. 1. Structure of a t-norm T on L_n defined as an ordinal sum. The white points represent where T takes the value of the minimum, while the black points represent where it takes the value of each T_i.

Proposition 3.5. *([1,5]) Let* $r \in L_n$, $J = \{0, 1, \dots, r\}$ *and* $T = \langle ([a_i, a_{i+1}], T_i)_{i\in J} \rangle$ *be an ordinal sum of t-norms on* $[a_i, a_{i+1}]$. *Then,* T *is a smooth t-norm on* L_n *if, and only if,* T_i *is smooth for all* $i \in J$.

Finally, a characterization of the smoothness of any t-norm by means of an ordinal sum structure is given.

Theorem 3.1. *([5]) A t-norm* T *on* L_n *is smooth if, and only if, there exists a natural number* r *with* $0 \leqslant r \leqslant n-1$ *and a subset* I *of* L_n, $I = \{0 = a_0 < a_1 < \cdots < a_r < a_{r+1} = n\}$, *such that* T *is given by*

$$T(x, y) = \begin{cases} \max\{a_i, x + y - a_{i+1}\}, & \text{if } (x, y) \in [a_i, a_{i+1}]^2,\ 0 \leqslant i \leqslant r, \\ \min\{x, y\}, & \text{otherwise.} \end{cases} \tag{5}$$

Remark 3.1. The structure shown in Theorem 3.1 has some relation with the case in [0, 1]: a t-norm T on [0, 1] is continuous if, and only if, it can be expressed as an ordinal sum of continuous Archimedean t-norms [19, Theorem 5.11]. Bearing this result in mind, together with the smoothness-continuity analogy, as there is only one Archimedean smooth t-norm on L_n, as stated in Proposition 3.3, the cases of ordinal sums in the L_n framework are condensed as ordinal sums of Łukasiewicz t-norms on L_n.

With the notation of Theorem 3.1, I represents the set of idempotent elements of T. From the given characterization, we have the following corollary:

Corollary 3.1.1. *([5]) The correspondence* $\psi : \mathcal{T}_{\text{div}} \to \mathcal{P}(L_n \backslash \{0, n\})$ *between the set of all smooth t-norms on* L_n *and the power set of* $\{1, 2, \dots, n-1\}$, *defined by* $\psi(T) = I \backslash \{0, n\}$ *is a bijection. Thus, there are exactly* 2^{n-1} *smooth t-norms on* L_n.

To end this section, we introduce the problem of the cardinality of t-norms on L_n. Evidently, in the [0, 1] framework the number of t-norms that can be constructed is infinite. In contrast, since t-norms on L_n are operators whose domain and image sets are finite, the number of these operators is also finite. Although for the case of smooth t-norms the cardinality is known thanks to Corollary 3.1.1, the general formula for cardinality of t-norms in L_n remains an open problem. By computation, the number of t-norms on L_n is known for $n \leqslant 13$, see [20]. More recently, the cardinality problem has been revisited in [21], dealing with the so-called maximal Archimedean t-norms on L_n and their count, finding that its number is governed by the Fibonacci sequence. Also, in [22] the number of some classes of t-norms defined on some special lattices with additional properties has been computationally determined.

3.2. Construction of t-norms

In [0, 1] there exist many construction methods for t-norms, among which we highlight the most important ones: using the pseudo-inverse of an increasing function; using additive and multiplicative generators of strictly decreasing or increasing functions, respectively; and using ordinal sums. For more information, see [19]. In the L_n case there are also construction methods, among which we highlight the discretization of t-norms on [0, 1] to L_n (adding additional properties) or considering also additive generators. Regarding the method of discretization of t-norms, we have the following results:

Proposition 3.6. *([20]) Let T be a t-norm defined on $[0,1]$ and $\Gamma_n = \left\{\frac{i}{n} \mid i \in L_n\right\} \subset [0,1]$. Then, for any finite chain L_n the restriction $D = n \cdot T|_{\Gamma_n^2}$ is a t-norm on L_n if, and only if, T satisfies $T(x,y) \in \{0, \min\{x,y\}\}$, for all $(x,y) \in [0,1]^2$.*

Conversely, we can also convert a t-norm on L_n into a t-norm on [0, 1]:

Proposition 3.7. *([19]) Let $n \in \mathbb{N}$ and let T be a t-norm on L_n. The binary operator $T' : [0,1]^2 \to [0,1]$ given by*

$$T'(x,y) = \begin{cases} \frac{1}{n} \cdot T\left(\lfloor nx \rfloor, \lfloor ny \rfloor\right), & \text{if } (x,y) \in [0,1)^2, \\ \min\{x,y\}, & \text{otherwise,} \end{cases} \tag{6}$$

is a t-norm on $[0,1]$ which satisfies $n \cdot T'\big|_{\Gamma_n^2} = T$.

The other technique we gather is the additive generator method. For the sake of clarity, we will define the concept of additive generator on [0, 1] introduced in the following definition in order to give the construction method later in Proposition 3.8.

Definition 3.3. ([19]) An *additive generator* $t : [0,1] \to [0,+\infty]$ of a t-norm T on $[0,1]$ is a strictly decreasing function which is right-continuous in 0 and it satisfies $t(1) = 0$, such that for all $(x,y) \in [0,1]^2$ we have

$$t(x) + t(y) \in \mathrm{Ran}(t) \cup [t(0), +\infty] \tag{7}$$

$$T(x,y) = t^{(-1)}(t(x) + t(y)) \tag{8}$$

where $t^{(-1)}$ denotes the pseudo-inverse[2] of t:

$$t^{(-1)}(y) = \sup\{x \in [0,1] \mid t(x) > y\}. \tag{9}$$

Proposition 3.8. *([19]) Let $n \in \mathbb{N}$, T be a t-norm on $[0,1]$ and $t : [0,1] \to [0,+\infty]$ be an additive generator of T.*

(i) If $t(0) = +\infty$, i.e., T is strictly monotone, the restriction $T|_{\Gamma_n^2}$ is a t-norm on L_n if, and only if, $n = 1$.
(ii) If $t(0) < +\infty$, the restriction $T|_{\Gamma_n^2}$ is a t-norm on L_n if, and only if, for all $i, j \in \{0, 1, \ldots, n\}$ we have

$$t\left(\frac{i}{n}\right) + t\left(\frac{j}{n}\right) \in \mathrm{Ran}\left(t|_{\Gamma_n}\right) \cup [t(0), +\infty]. \tag{10}$$

We have just shown in Definition 3.3 and Proposition 3.8 how from an additive generator in [0, 1] one can generate a t-norm in [0, 1] and restrict it to L_n. However, the study performed in [23] directly allows to consider generators in L_n. Let $\mathbb{R}^+$ be the set of positive real numbers and let $\mathcal{F}$ be the class of strictly decreasing functions $f : L_n \to \mathbb{R}^+$ such that $f(n) = 0$. With these conditions, f is a generator of conjunctions on L_n.

---

[2] We recommend the reader to visit [19], where the general definition of pseudo-inverse is presented. In this case, the expression is simpler due to the strictly decreasingness of the function.

Proposition 3.9. *([23]) Given $f \in \mathcal{F}$, the binary operation $C : L_n \times L_n \to L_n$, defined by*

$$C(i, j) = f^{(-1)}\left(f(i) + f(j)\right), \qquad i, j \in L_n, \tag{11}$$

is a conjunction on L_n, where $f^{(-1)}$ denotes the pseudo-inverse of f:

$$f^{(-1)}(t) = \min\{i \in L_n \mid f(i) \leqslant t\}. \tag{12}$$

In this case, it is denoted by $C = \langle f \rangle$ to indicate that the conjunction C is defined from f via Equation (11).

We say that an additive generator f is associative if the conjunction given by $C = \langle f \rangle$ is associative and, thus, it is a t-norm. When other properties such as concavity and convexity (see [23] for more information) are imposed on the additive generator f, the Archimedean and smooth properties are obtained.

Proposition 3.10. *([23]) Let $f : L_n \to \mathbb{R}$ be an additive generator. The following statements hold:*

- *If f is concave, the conjunction C on L_n additively generated by f has only 0 and n as idempotent elements.*
- *If f is convex, the conjunction C on L_n additively generated by f is smooth.*

3.3. Triangular conorms: duality from triangular norms

In Definition 3.1 the definition of t-norm has been formulated as a particular case of conjunction. Now, we will define a particular case of disjunction, known as t-conorm.

Definition 3.4. ([5]) A *triangular conorm on L_n* (briefly *t-conorm on L_n*) is a binary mapping $S : L_n \times L_n \to L_n$ such that it is an associative and commutative disjunction with 0 as neutral element; that is, $S(0, x) = x$ for all $x \in L_n$.

Making an analogy with the t-norms presented in Example 3.1, the following example shows some well-known t-conorms.

Example 3.2. Some examples of t-conorms are:

- The maximum t-conorm:

$$S_{\boldsymbol{M}}(x, y) = \max\{x, y\}.$$

- The drastic t-conorm:

$$S_{\boldsymbol{D}}(x, y) = \begin{cases} n, & \text{if } x, y \neq 0, \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

- The nilpotent maximum t-conorm:

$$S_{\boldsymbol{nM}}(x, y) = \begin{cases} n, & \text{if } x + y \geqslant n, \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

- The Łukasiewicz t-conorm:

$$S_{\boldsymbol{L}}(x, y) = \min\{x + y, n\}.$$

In Proposition 2.1 the negation N_C has been presented as the only strong negation on L_n, acting as the only decreasing automorphism. The importance of this operator is presented in the following theorem, under the concept of N-duality.

Theorem 3.2. *([5]) Let $\mathcal{T}(L_n)$ and $\mathcal{S}(L_n)$ be the set of all t-norms and t-conorms on L_n, respectively. Given the strong negation N_C on L_n, a bijective correspondence from $\mathcal{T}(L_n)$ to $\mathcal{S}(L_n)$ can be considered: for each $T \in \mathcal{T}(L_n)$, the binary operator T_N given by*

$$T_N(x, y) = n - T(n - x, n - y) \qquad \text{for all } x, y \in L_n \tag{13}$$

is a t-conorm on L_n, called the N-dual t-conorm of T. In a similar way, for each $S \in \mathcal{S}(L_n)$, the binary operator S_N given by

$$S_N(x, y) = n - S(n - x, n - y) \qquad \text{for all } x, y \in L_n \tag{14}$$

is a t-norm on L_n, called the N-dual t-norm of S.

A dual pair (T, S) is a t-norm T and a t-conorm S such that $S = T_N$ or $T = S_N$; in this case, the triplet (T, S, N_C) is often called *De Morgan Triplet*. It can be easily deduced from Equations (13) and (14) that $(T_N)_N = T$ and $(S_N)_N = S$.

The fundamental property offered by Theorem 3.2 is the bijection between $\mathcal{T}(L_n)$ and $\mathcal{S}(L_n)$. This means that all the properties mentioned above, such as Propositions 3.1, 3.2, 3.3 and 3.5 have their equivalent for t-conorms (naturally, correctly adapting the conditions for these operators). For example, smoothness is also characterized by formulating the equivalent result of Theorem 3.1 for t-conorms.

Theorem 3.3. *([5]) A t-conorm S on L_n is smooth if, and only if, there exists a natural number r with $0 \leqslant r \leqslant n - 1$ and a subset I of L_n, $I = \{0 = a_0 < a_1 < \cdots < a_r < a_{r+1} = n\}$, such that S is given by*

$$S(x, y) = \begin{cases} \min\{a_{i+1}, x + y - a_i\}, & \text{if } (x, y) \in [a_i, a_{i+1}]^2,\ 0 \leqslant i \leqslant r, \\ \max\{x, y\}, & \text{otherwise.} \end{cases} \tag{15}$$

Finally, we present the relation between a dual pair (T, S) and the property of divisibility for t-norms and t-conorms given by Frank's equation:

Proposition 3.11. *([5]) A pair (T, S), where T and S are a t-norm and a t-conorm on L_n, respectively, is a solution of the functional equation*

$$T(x, y) + S(x, y) = x + y, \qquad \text{for all } x, y \in L_n, \tag{16}$$

if, and only if, T and S are divisible with the same set of idempotent elements.

4. Uninorms

In this section we will deal with another kind of aggregation functions: uninorms on L_n. These operators were introduced as a generalization of t-norms and t-conorms on L_n based on the fact that these two classes of aggregation functions are defined by means of the same three axioms just differing in the value of the neutral element [24]. From now on, we denote by $\mathcal{U}$ the set of uninorms defined on L_n.

Definition 4.1. ([25,26]) A *uninorm on L_n* is a binary operation $U : L_n \times L_n \to L_n$ which is commutative, associative and increasing in each argument, with neutral element $e \in L_n$, i.e., for all $x \in L_n$ it holds that $U(x, e) = U(e, x) = x$.

According to this definition, a uninorm on L_n becomes a t-norm when $e = n$ and a t-conorm when $e = 0$. Otherwise, a uninorm is said to be a proper uninorm if its neutral element satisfies $0 < e < n$. In addition, for any uninorm U it holds that $U(0, n) = U(n, 0) \in \{0, n\}$ (see [26]); therefore uninorms are classified as conjunctives when $U(0, n) = 0$ and disjunctives when $U(0, n) = n$.

A property that has been also studied in the context of uninorms is smoothness. From the studies [2,27], it is known that in the [0, 1] framework there are no continuous proper uninorms. In the case of uninorms on L_n, the following result indicates that there are no smooth proper uninorms.

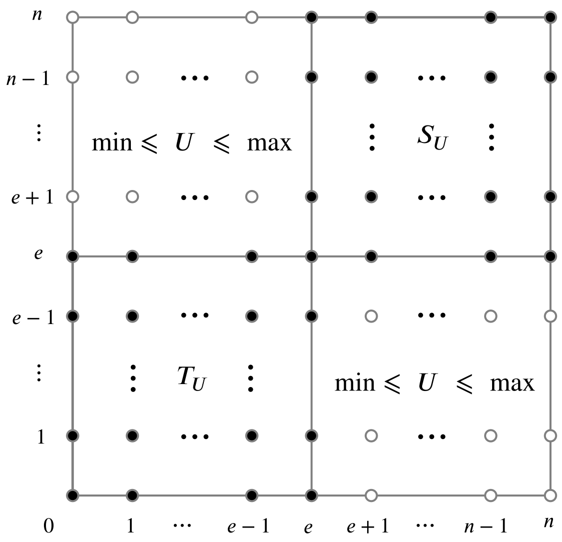

Fig. 2. Structure of any uninorm U on L_n with neutral element e. The black dots represent the points where U is equal to T_U or S_U, while the white dots represent the values where U satisfies $\min\{x, y\} \leqslant U(x, y) \leqslant \max\{x, y\}$, for all $(x, y) \in C_e$.

Proposition 4.1. *([2,26]) Let U be a uninorm on L_n. Then, U is smooth if, and only if, it is a smooth t-norm or a smooth t-conorm on L_n.*

To conclude the first part of the section, we now introduce the concept of duality of a uninorm with respect to the unique strong negation.

Definition 4.2. ([26]) Let U be a uninorm on L_n and N_C be the unique strong negation on L_n. Then, the *N-dual operator* of U is given by

$$U_N(x, y) = N\left(U\left(N(x), N(y)\right)\right), \qquad x, y \in L_n. \tag{17}$$

Proposition 4.2. *([26]) Let U be a uninorm on L_n and N_C be the unique strong negation on L_n. Then, the N-dual of U is also a uninorm on L_n with $n - e$ as neutral element. Moreover, if U is conjunctive (disjunctive), U_N is disjunctive (conjunctive), and vice versa.*

Given any proper uninorm U, it can be described according to [25], as follows: its restriction to the subdomain $[0, e]^2$ is a t-norm T_U on $[0, e]$, while its restriction to $[e, n]^2$ is a t-conorm S_U on $[e, n]$, see Fig. 2; we introduce the notation $U = \langle T_U, e, S_U\rangle$ to represent this behavior. In the remaining domain, denoted by

$$C_e = [0, e - 1] \times [e + 1, n] \cup [e + 1, n] \times [0, e - 1],$$

it holds that $\min\{x, y\} \leqslant U(x, y) \leqslant \max\{x, y\}$, for all $(x, y) \in C_e$.

4.1. Uninorms on $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$

Due to the inequality $\min\{x, y\} \leqslant U(x, y) \leqslant \max\{x, y\}$, for all $(x, y) \in C_e$, the first attempt to generate uninorms is to take always the minimum or the maximum in C_e. We denote by $\mathcal{U}_{\min}$ the class of uninorms on L_n that in C_e behave as the minimum, and by $\mathcal{U}_{\max}$ the class of uninorms on L_n that behave as the maximum. In other words,

$$\mathcal{U}_{\min} = \{U \in \mathcal{U} \mid U(x, y) = \min\{x, y\},\ (x, y) \in C_e\},$$
$$\mathcal{U}_{\max} = \{U \in \mathcal{U} \mid U(x, y) = \max\{x, y\},\ (x, y) \in C_e\}.$$

As it has been shown in [26], all operators belonging to $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are still uninorms.

As in the $[0, 1]$ framework, the partial mappings $U(x, n)$ and $U(x, 0)$, with $x \in L_n$, play an important role in determining the values of U in C_e. The following theorem, which has been adapted from its corresponding version in $[0, 1]$, remains true for uninorms on L_n.

Theorem 4.1. *([27]) Let U be a proper uninorm on L_n with neutral element e.*

1. *If $U(0, n) = 0$ and the section $U(x, n)$ is smooth except on $x = e$, then U is given by*

$$U(x,y)=\begin{cases} T(x,y), & \text{if } 0 \le x \le e \text{ and } 0 \le y \le e,\\ e+S(x-e,y-e), & \text{if } e \le x \le n \text{ and } e \le y \le n,\\ \min\{x,y\}, & \text{if } (x,y) \in C_e,\end{cases} \tag{18}$$

and therefore $U \in \mathcal{U}_{\min}$.

2. *If $U(0, n) = n$ and the section $U(x, 0)$ is smooth except on $x = e$, then U is given by*

$$U(x,y)=\begin{cases} T(x,y), & \text{if } 0 \le x \le e \text{ and } 0 \le y \le e,\\ e+S(x-e,y-e), & \text{if } e \le x \le n \text{ and } e \le y \le n,\\ \max\{x,y\}, & \text{if } (x,y) \in C_e,\end{cases} \tag{19}$$

and therefore $U \in \mathcal{U}_{\min}$.

In both expressions, T is a t-norm on $[0, e]$ and S is a t-conorm on $[0, n - e]$.

4.2. Idempotent uninorms

In this section we will deal with the class of idempotent uninorms on L_n, denoted by $\mathcal{U}_{\text{ide}}$; that is, those which $U(x, x) = x$ for all $x \in L_n$. The characterization of this family of uninorms has been performed mainly from three different points of view. First, by studying decreasing unary functions satisfying certain symmetry properties; secondly, having some common ground with the previous scheme, using a concept similar to symmetry, known simply as property $(\tilde{\text{S}})$; and finally, from the perspective of isolated points.

Making use of decreasing unary functions $g : L_n \to L_n$, let us begin by giving the definition of its associated completed graph, as well as the concept of Id-symmetry.

Definition 4.3. ([25]) Given any decreasing function $g : L_n \to L_n$, its *completed graph* F_g is defined as the subset of L_n^2 given by

$$F_g = (\{0\} \times [g(0), n]) \cup (\{n\} \times [0, g(n)]) \cup \{(x, y) \in [0, n-1] \times [0, n] \mid g(x+1) \leqslant y \leqslant g(x)\}. \tag{20}$$

Definition 4.4. ([25]) A *subset* F of L_n^2 is said to be *Id-symmetrical* if for all $(x, y) \in L_n^2$, it holds that $(x, y) \in F$ if, and only if, $(y, x) \in F$.

Definition 4.5. ([25]) A *decreasing function* $g : L_n \to L_n$ is said to be *Id-symmetrical* if its completed graph F_g is Id-symmetrical.

Example 4.1. Let us consider the decreasing function $g : L_6 \to L_6$ given by

$$g(x)=\begin{cases} 6, & \text{if } x \in \{0,1\},\\ 5, & \text{if } x = 2,\\ 2, & \text{if } x \in \{3,4,5\},\\ 0, & \text{if } x = 6.\end{cases} \tag{21}$$

This function is not Id-symmetrical, and its completed graph is given by the set of points

$$F_g = \{(0,6), (1,6), (1,5), (2,5), (2,4), (2,3), (2,2), (3,2), (4,2), (5,2), (5,1), (5,0), (6,0)\}.$$

As can be seen in Fig. 3, F_g is also not Id-symmetrical.

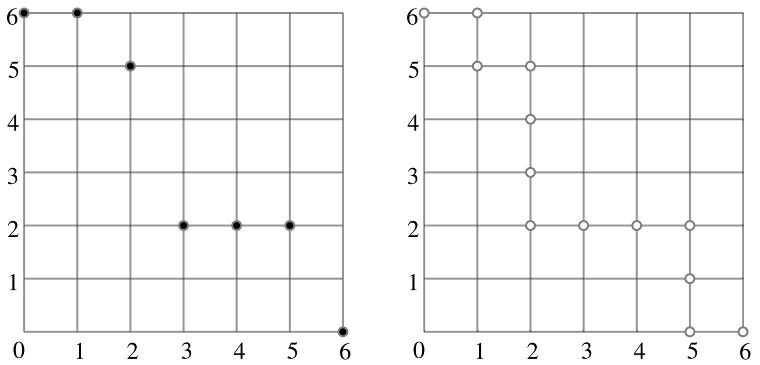

Fig. 3. Plot of the function g formulated in Equation (21) (left) and its completed graph F_g (right).

For more information on the study of the symmetry of g and F_g, we encourage the reader to visit [25,28]. Next, we recall the last result before providing the characterization of idempotent uninorms, which gives a way to construct a symmetric extension from a decreasing function g.

Proposition 4.3. *([25]) Let $e \in L_n$ such that $0 < e < n$ and let $g : [0, e] \to [e, n]$ be a decreasing function such that $g(e) = e$. Then there exists exactly one Id-symmetrical extension of g, $\overline{g} : L_n \to L_n$, given by*

$$\overline{g}(x) = \begin{cases} g(x), & \text{if } x \leqslant e, \\ \max\{z \in [0, e] \mid g(z) \geqslant x\}, & \text{if } e \leqslant x \leqslant g(0), \\ 0, & \text{if } x > g(0). \end{cases} \tag{22}$$

Theorem 4.2. *([25]) A binary operation U on L_n with neutral element $0 < e < n$ is an idempotent uninorm on L_n if, and only if, there exists a decreasing function $g : [0, e] \to [e, n]$ with $g(e) = e$ such that*

$$U(x, y) = \begin{cases} \min\{x, y\}, & \text{if } y \leqslant \overline{g}(x) \text{ and } x \leqslant \overline{g}(0), \\ \max\{x, y\}, & \text{elsewhere,} \end{cases} \tag{23}$$

where $\overline{g}$ is the unique Id-symmetrical extension of g.

Although at first glance the existence of such a function g satisfying the conditions of Theorem 4.2 may seem difficult to prove, the authors of [25] provided a way to compute the expression of g from an idempotent uninorm. Indeed, given an idempotent uninorm U with neutral element $0 < e < n$, g can be obtained as follows:

$$g(x) = \begin{cases} \max\{z \in L_n \mid U(x, z) = \min\{x, z\}\}, & \text{if } x \leqslant e, \\ 0, & \text{if } x > e \text{ and } \min\{z \in L_n \mid U(x, z) = \max\{x, z\}\} = 0, \\ \min\{z \in L_n \mid U(x, z) = \max\{x, z\}\} - 1, & \text{otherwise.} \end{cases} \tag{24}$$

As a consequence of the bijective correspondence established in the theorem above between decreasing functions $g : [0, e] \to [e, n]$ with $g(e) = e$ and idempotent uninorms with neutral element $0 < e < n$, we can easily count the total number of such operators.

Theorem 4.3. *([25]) The following statements hold:*

1. *The number of idempotent uninorms on L_n, with $n \geqslant 2$, with neutral element $e \in L_n$, is given by*

$$I_{e,n} = \binom{n}{e}.$$

2. *The total number of idempotent uninorms on L_n, with $n \geqslant 2$, is given by*

$$I_n = \sum_{e=0}^{n} I_{e,n} = 2^n.$$

Following the second point of view about the different techniques aforementioned, De Baets et al. have recently proposed in [29] a characterization using a symmetry-related property for unary functions defined on L_n, called property ($\tilde{\mathrm{S}}$).

Definition 4.6. ([29]) A unary function $f : L_n \to L_n$ is said to have *property* ($\tilde{S}$) if $0 < y \leqslant f(x)$ implies $x \leqslant f(y)$, for all $x, y \in L_n$.

Theorem 4.4. *([29]) A binary operation U on L_n with neutral element $0 < e < n$ is an idempotent uninorm if, and only if, there exists a unique decreasing unary function $g : L_n \to L_n$ with fixed point e and property ($\tilde{S}$) such that U is given by*

$$U(x, y) = \begin{cases} \min\{x, y\}, & \textit{if } y \leqslant g(x) \textit{ and } x \leqslant g(0), \\ \max\{x, y\}, & \textit{otherwise.} \end{cases} \tag{25}$$

Despite the fact that it may also seem difficult to prove the existence of a function g satisfying the conditions of Theorem 4.4, the authors give a way to construct it from an idempotent uninorm U, given by

$$g(x) = \sup\{y \in L_n \mid U(x, z) = \min\{x, z\} \text{ for all } z \in \{w \in L_n \mid w \leqslant y\}\},$$

for all $x \in L_n$, assuming $\sup \emptyset = 0$. With this expression, g satisfies property ($\tilde{\mathrm{S}}$) [29].

To conclude the study on idempotent uninorms, another technique for their characterization comes from [13,30]. In these papers, based on the concept of isolated point (which we will not go into in detail), a characterization is proposed which, interestingly, does not require the property of isolation in the conditions; it is only used in the proof of the result.

Theorem 4.5. *([13,30]) A binary operation $F : L_n \times L_n \to L_n$ is conservative, commutative and increasing in each argument if, and only if, it is an idempotent uninorm on L_n.*

4.3. Uninorms with smooth underlying operators

Another class worthy to be studied is the class of uninorms on L_n with smooth underlying operators, which will be denoted by $\mathcal{U}_{\mathrm{sts}}$. Its structure is fully characterized unlike the case [0, 1], where the full characterization is only known when the underlying operators are continuous and Archimedean; apart from this, other particular cases have also been characterized, see [2] for more information. The following theorem represents the structure of $\mathcal{U}_{\mathrm{sts}}$.

Theorem 4.6. *([28]) A binary operator U on L_n with neutral element $0 < e < n$ and idempotent elements $J = \{0 = a_0 < a_1 < \cdots < a_r = e = b_0 < b_1 < \cdots < b_s = n\}$ is a uninorm on L_n with smooth underlying operators T_U and S_U if, and only if, there exists a decreasing function $g : [0, e] \to [e, n]$ with fixed point e that satisfies the following conditions:*

1. *for all $a_i \in J$, there exists $b_j \in J$ such that $g(a_i) = b_j$,*
2. *if $x \in [a_i, a_{i+1}[$, then $g(x) = g(a_i)$,*

and U is given by

$$U(x, y) = \begin{cases} T_U(x, y), & \textit{if } (x, y) \in [0, e]^2, \\ S_U(x, y), & \textit{if } (x, y) \in [e, n]^2, \\ \min\{x, y\}, & \textit{if } (x, y) \in C_e,\ y \leqslant \overline{g}(x) \textit{ and } x \leqslant \overline{g}(0), \\ \max\{x, y\}, & \textit{otherwise,} \end{cases} \tag{26}$$

where $\overline{g}$ *is the only Id-symmetric extension of* g.

From this general result, it can be concluded that in the L_n domain every uninorm U with smooth underlying operators is internal in C_e; that is, $U(x, y) \in \{x, y\}$ for all $(x, y) \in C_e$. The set of uninorms that are internal in C_e will be denoted by $\mathcal{U}_{\text{int}}$. Further, some conclusions obtained in [2] can be drawn depending on the smooth t-norm and t-conorm considered. For instance,

- When $T_U = T_{\mathbf{M}}$ and $S_U = S_{\mathbf{M}}$, all the elements are idempotent, and conditions 1 and 2 in Theorem 4.6 are always fulfilled.
- When $T_U = T_{\mathbf{L}}$ and $S_U = S_{\mathbf{L}}$, the only idempotent elements are 0, e and n, and only two possible functions g can be constructed (see Proposition 4.4 below).
- When $T_U = T_{\mathbf{M}}$ and $S_U = S_{\mathbf{L}}$, the function g must be such that $g(x) \in \{e, n\}$ for all $x \in [0, e[$ (see Proposition 4.5 below).
- When $T_U = T_{\mathbf{L}}$ and $S_U = S_{\mathbf{M}}$, the case is dual to the previous one (see Proposition 4.6 below).

Proposition 4.4. *([28]) There are exactly two uninorms on* L_n*, with* $n \geqslant 2$ *and neutral element* $0 < e < n$*, such that their underlying operators are* $T_U = T_{\mathbf{L}}$ *and* $S_U = S_{\mathbf{L}}$*:*

- *If* U *is conjunctive, then* $U = \langle T_{\mathbf{L}}, e, S_{\mathbf{L}} \rangle$ *and* $U \in \mathcal{U}_{\min}$.
- *If* U *is disjunctive, then* $U = \langle T_{\mathbf{L}}, e, S_{\mathbf{L}} \rangle$ *and* $U \in \mathcal{U}_{\max}$.

Proposition 4.5. *([28]) Let* U *be a uninorm on* L_n *with neutral element* $0 < e < n$ *and underlying operators given by* $T_U = T_{\mathbf{M}}$ *and* $S_U = S_{\mathbf{L}}$*. Then, there exists* $r \in [0, e]$ *such that* U *is given by*

$$U(x, y) = \begin{cases} S_{\mathbf{L}}(x, y), & \text{if } (x, y) \in [e, n]^2, \\ \max\{x, y\}, & \text{if } (x, y) \in [r, e-1] \times [e+1, n] \text{ or } (x, y) \in [e+1, n] \times [r, e-1], \\ \min\{x, y\}, & \text{otherwise.} \end{cases} \tag{27}$$

Therefore, when $n \geqslant 2$ *there are exactly* $e + 1$ *possible uninorms of this type.*

Proposition 4.6. *([28]) Let* U *be a uninorm on* L_n *with neutral element* $0 < e < n$ *and underlying operators given by* $T_U = T_{\mathbf{L}}$ *and* $S_U = S_{\mathbf{M}}$*. Then, there exists* $r \in [e, n]$ *such that* U *is given by*

$$U(x, y) = \begin{cases} T_{\mathbf{L}}(x, y), & \text{if } (x, y) \in [0, e]^2, \\ \min\{x, y\}, & \text{if } (x, y) \in [0, e-1] \times [e+1, r] \text{ or } (x, y) \in [e+1, r] \times [0, e-1], \\ \max\{x, y\}, & \text{otherwise.} \end{cases} \tag{28}$$

Therefore, when $n \geqslant 2$ *there are exactly* $n - e + 1$ *possible uninorms of this type.*

4.4. Summary of families of uninorms

By way of conclusion, in this section we survey and classify the different families of uninorms presented in Sections 4.1, 4.2 and 4.3. Proposition 4.7 establishes the links between different classes of uninorms, while Fig. 4 graphically schematizes these relationships.

Proposition 4.7. *In the class of all uninorms* $\mathcal{U}$ *defined on* L_n*, it is satisfied:*

1. $\mathcal{U}_{\text{ide}} \subset \mathcal{U}_{\text{sts}} \subset \mathcal{U}_{\text{int}}$.
2. $\mathcal{U}_{\min}, \mathcal{U}_{\max} \subset \mathcal{U}_{\text{int}}$.
3. $\mathcal{U}_{\text{ide}} \cap \mathcal{U}_{\min} \neq \emptyset$ *and* $\mathcal{U}_{\text{ide}} \cap \mathcal{U}_{\max} \neq \emptyset$.
4. $(\mathcal{U}_{\text{sts}} \backslash \mathcal{U}_{\text{ide}}) \cap \mathcal{U}_{\min} \neq \emptyset$ *and* $(\mathcal{U}_{\text{sts}} \backslash \mathcal{U}_{\text{ide}}) \cap \mathcal{U}_{\max} \neq \emptyset$.

The diagrams in Fig. 4 are well defined, in the sense that the inclusions are strict and therefore do not collapse into any other class, as shown in the next example.

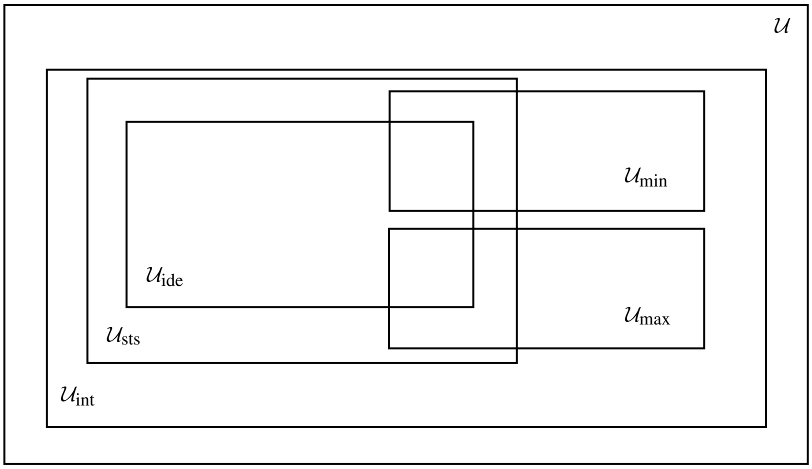

Fig. 4. Classification diagram of the different families of uninorms on L_n.

Example 4.2. Let us consider the following operators:

$$U_1 = \begin{array}{cc} \begin{array}{c}4\\3\\2\\1\\0\end{array} & \begin{bmatrix} 0&2&2&4&4\\ 0&1&2&3&4\\ 0&0&0&2&2\\ 0&0&0&1&2\\ 0&0&0&0&0 \end{bmatrix} \\ & \begin{array}{ccccc}0&1&2&3&4\end{array} \end{array}, \quad U_2 = \begin{array}{cc} \begin{array}{c}4\\3\\2\\1\\0\end{array} & \begin{bmatrix} 4&4&4&4&4\\ 0&1&2&3&4\\ 0&0&0&2&4\\ 0&0&0&1&4\\ 0&0&0&0&4 \end{bmatrix} \\ & \begin{array}{ccccc}0&1&2&3&4\end{array} \end{array}, \quad U_3 = \begin{array}{cc} \begin{array}{c}4\\3\\2\\1\\0\end{array} & \begin{bmatrix} 0&1&4&4&4\\ 0&1&2&3&4\\ 0&1&2&2&4\\ 0&0&1&1&1\\ 0&0&0&0&0 \end{bmatrix} \\ & \begin{array}{ccccc}0&1&2&3&4\end{array} \end{array},$$

$$U_4 = \begin{array}{cc} \begin{array}{c}4\\3\\2\\1\\0\end{array} & \begin{bmatrix} 4&4&4&4&4\\ 0&1&2&3&4\\ 0&0&1&2&4\\ 0&0&0&1&4\\ 0&0&0&0&4 \end{bmatrix} \\ & \begin{array}{ccccc}0&1&2&3&4\end{array} \end{array}, \quad U_5 = \begin{array}{cc} \begin{array}{c}4\\3\\2\\1\\0\end{array} & \begin{bmatrix} 0&4&4&4&4\\ 0&1&2&3&4\\ 0&1&2&2&4\\ 0&1&1&1&4\\ 0&0&0&0&0 \end{bmatrix} \\ & \begin{array}{ccccc}0&1&2&3&4\end{array} \end{array}, \quad U_6 = \begin{array}{cc} \begin{array}{c}4\\3\\2\\1\\0\end{array} & \begin{bmatrix} 0&1&2&4&4\\ 0&1&2&3&4\\ 0&0&0&2&2\\ 0&0&0&1&1\\ 0&0&0&0&0 \end{bmatrix} \\ & \begin{array}{ccccc}0&1&2&3&4\end{array} \end{array}.$$

All of the above U_i are uninorms on L_4 with neutral element $e = 3$, different underlying t-norms on L_3 and $S_{\mathbf{M}}$ as the underlying t-conorm on L_2. The following statements hold:

- $U_1 \notin \mathcal{U}_{\text{int}}$.
- $U_2 \in \mathcal{U}_{\text{int}}$ but $U_2 \notin \mathcal{U}_{\text{sts}}$.
- $U_3 \in \mathcal{U}_{\text{sts}}$ but $U_3 \notin \mathcal{U}_{\text{ide}}$, $U_3 \notin \mathcal{U}_{\max}$ and $U_3 \notin \mathcal{U}_{\min}$.
- $U_4 \in \mathcal{U}_{\text{sts}} \cap \mathcal{U}_{\max}$ but $U_4 \notin \mathcal{U}_{\text{ide}}$.
- $U_5 \in \mathcal{U}_{\text{sts}} \cap \mathcal{U}_{\text{ide}}$ but $U_5 \notin \mathcal{U}_{\min} \cup \mathcal{U}_{\max}$.
- $U_6 \notin \mathcal{U}_{\text{sts}}$ and $U_6 \in \mathcal{U}_{\min}$.

5. Copulas

The concept of copula on [0, 1] first appeared in [31] in the framework of theoretical statistics. Without going into details, the concept of copula has two aspects. On the one hand, it refers to functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions via the well-known Sklar's Theorem, that was first proved in [32]; on the other hand, they are distribution functions whose one-dimensional margins are uniform and therefore provide a way of studying scale-free measures of dependence [33]. In the L_n domain, the definition is the following one:

Definition 5.1. ([34]) A *copula* C on L_n is a binary operator $C : L_n \times L_n \to L_n$ that satisfies the 2-increasingness condition, 0 is an absorbing element and n is a neutral element of C; i.e., it satisfies the following axioms:

(C1) $C(x_1, y_1) + C(x_2, y_2) \geqslant C(x_1, y_2) + C(x_2, y_1)$, whenever $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ (2-increasingness).
(C2) $C(x, 0) = C(0, x) = 0$, for all $x \in L_n$.
(C3) $C(x, n) = C(n, x) = x$, for all $x \in L_n$.

Remark 5.1. On the one hand, in Axiom (C1) in Definition 5.1, the sum is understood as the usual sum between natural numbers. Although the result on each side could be out of bounds of L_n, the axiom is well defined, verifiable and analogous to the 2-increasingness of copulas in [0, 1] (the result on each side could be out of bounds of [0, 1]). On the other hand, the fact that the result of the sum could be out of bounds may be understood as a contradiction with the fact that connectives on L_n allow to handle qualitative information directly without any numerical conversion. Indeed, the values out of bounds of L_n would not have a direct translation into the qualitative set. For this reason, we propose the following definition:

Definition 5.1*. A *copula* C on L_n is a binary operator $C : L_n \times L_n \to L_n$ that satisfies the 2-increasingness condition, 0 is an absorbing element and n is a neutral element of C; i.e., it satisfies the following axioms:

(C1*) C is increasing in both arguments and then $C(x_2, y_2) - C(x_2, y_1) \geqslant C(x_1, y_2) - C(x_1, y_1)$, whenever $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.
(C2) $C(x, 0) = C(0, x) = 0$, for all $x \in L_n$.
(C3) $C(x, n) = C(n, x) = x$, for all $x \in L_n$.

Note that Axiom (C1*) is equivalent to Axiom (C1) when Axioms (C2) and (C3) hold. Moreover, in (C1*) the condition $C(x_2, y_2) - C(x_2, y_1) \geqslant C(x_1, y_2) - C(x_1, y_1)$ is only checked when C is increasing in both arguments, ensuring that both sides of the inequality are within the bounds of L_n.

Note that if we add associativity and commutativity to a copula, then it becomes a t-norm on L_n, since 2-increasingness condition is a generalization of the increasingness property in each argument. Therefore, some examples of copulas will be given by t-norms, as it will be seen in Example 5.1. However, not every copula is a t-norm.

Example 5.1. Some examples of copulas are:

- The minimum t-norm ($T_{\boldsymbol{M}}$, denoted also by $C_{\boldsymbol{M}}$) and the Łukasiewicz t-norm ($T_{\boldsymbol{L}}$ denoted also by $C_{\boldsymbol{L}}$), are also copulas defined on L_n.
- The operator in L_3 given by

$$C(x, y) = \begin{cases} C_{\boldsymbol{L}}(x, y) = \max\{0, x + y - 3\}, & \text{if } (x, y) \neq (1, 2), \\ 1, & \text{if } (x, y) = (1, 2), \end{cases} \tag{29}$$

  is a copula on L_3 which is not a t-norm since it is not associative.

Next, we summarize some basic properties of copulas.

Proposition 5.1. *([34,35]) Let C be a copula on L_n. The following statements hold:*

- $C_{\mathbf{L}} \leqslant C \leqslant C_{\mathbf{M}}$.
- *C is increasing in each argument.*
- *C satisfies the Lipschitz condition with constant 1.*
- *The only copula C on L_n that satisfies $C(x, x) = x$, for all $x \in L_n$, is $C = C_{\mathbf{M}}$.*
- *The only copula C on L_n that satisfies $C(x, n - x) = 0$ or $C(n - x, x) = 0$, for all $x \in L_n$, is $C = C_{\mathbf{L}}$.*

- *The only associative copula on L_n with 0 and n as the only idempotent elements is $C_{\mathbf{L}}$.*
- *If a copula on L_n is associative, then it is commutative.*

To conclude this section, considering the second part of [34] and using some properties of copulas on L_n, a version of Sklar's theorem is proposed in [36] for these operators.

Theorem 5.1. *([36]) Let $F, G : \mathbb{R} \to [0,1]$ be distribution functions with $\operatorname{Ran} F, \operatorname{Ran} G \subseteq \Gamma_n$. If C is a copula on L_n, then*

$$H(x,y) = \frac{C(n \cdot F(x), n \cdot G(y))}{n} \tag{30}$$

for all $(x,y) \in \mathbb{R}^2$ is a joint distribution function with $\operatorname{Ran} H \subset \Gamma_n$ having F and G as marginal distribution functions. Conversely, if H is a joint distribution function with marginal distribution functions F and G such that $\operatorname{Ran} H \subseteq \Gamma_n$, there exists a copula C on L_n such that H is given by Equation (30). Moreover, this copula C is unique if, and only if, $\operatorname{Ran} F = \operatorname{Ran} G = \Gamma_n$.

5.1. Structure of copulas

As well as t-norms and t-conorms, copulas also accept a representation in terms of ordinal sums. But more than that, they can be characterized in terms of another structure that is less common in the research field of connectives defined on L_n: permutation matrices. Let us begin by studying the characterization by means of this type of matrices. From now on, we will denote by $\mathfrak{S}_n$ the group of permutations of n elements.

Definition 5.2. A *permutation matrix* is an $n \times n$ matrix such that there exists a permutation $\sigma \in \mathfrak{S}_n$ of L_n such that

$$a_{ij} = \begin{cases} 1, & \text{if } i = \sigma(j), \\ 0, & \text{otherwise,} \end{cases} \tag{31}$$

for all $i, j \in \{1, \ldots, n\}$.

The importance of these matrices lies in the following proposition, allowing any copula to be expressed as combinations of elements of a permutation matrix.

Proposition 5.2. *([34]) Let $n \geqslant 1$, and let $C : L_n^2 \to L_n$ be a binary operator. Then, C is a copula on L_n if, and only if, there exists an $n \times n$ permutation matrix $A = (a_{ij})$ such that, for all $(x,y) \in L_n^2$,*

$$C(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \sum_{i=1}^{x} \sum_{j=1}^{y} a_{ij}, & \text{otherwise.} \end{cases} \tag{32}$$

Unlike in the L_n case, in the $[0,1]$ domain copulas have a very different characterization from that provided by Proposition 5.2. Specifically, by so-called Dini derivatives, see [37, Theorem 1.6.8] for more information.

Example 5.2. Let us consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix} \in \mathfrak{S}_5$. The associated permutation matrix is

$$A_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

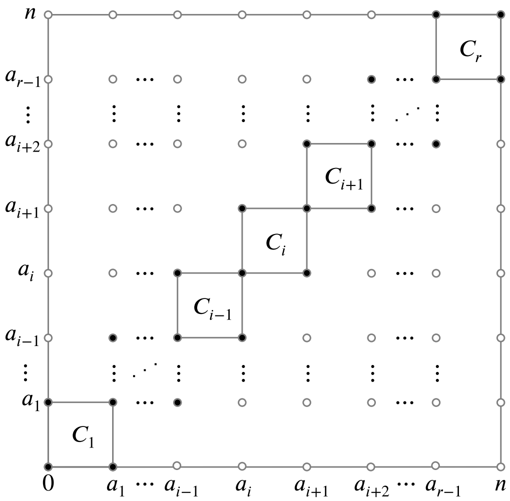

Fig. 5. Structure of a copula on L_n defined as an ordinal sum. The white points represent where it takes the value of the minimum, while the black points represent where it takes the value of the corresponding C_i.

Applying Proposition 5.2, the copula on L_5 generated from A_σ is given by

$$C = \begin{array}{c|cccccc|} 5 & 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 0 & 0 & 1 & 2 & 3 & 4 \\ 3 & 0 & 0 & 1 & 2 & 2 & 3 \\ 2 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

which is commutative.

A copula C given through a permutation matrix A by Equation (32) will be denoted by C_A. Proposition 5.2 defines a bijection between the set of $n \times n$ permutation matrices and the set of copulas; thus, there are $n!$ copulas on L_n. The following proposition shows the relation between commutative copulas and their permutation matrices.

Proposition 5.3. *([34]) A copula on L_n is commutative if, and only if, its associated matrix is symmetric.*

The other representation of the copula structure is by means of ordinal sums, analogously to the structure of t-norms seen in Definition 3.2.

Proposition 5.4. *([38]) Let C be a copula on L_n with the set of idempotent elements $I = \{0 = a_0, a_1, \ldots, a_{r-1}, a_r = n\}$ and let $J = \{1, \ldots, r\}$. Then C is an ordinal sum $C = \langle([a_{i-1}, a_i], C_i)_{i \in J}\rangle$; that is, C is given by*

$$C(x, y) = \begin{cases} C_i(x, y), & \text{if } (x, y) \in [a_{i-1}, a_i]^2 \text{ for some } i \in J, \\ \min\{x, y\}, & \text{otherwise,} \end{cases} \tag{33}$$

where C_i is a copula on $[a_{i-1}, a_i]$ for each $i \in J$.

Fig. 5 illustrates the structure of copulas given by Proposition 5.4.

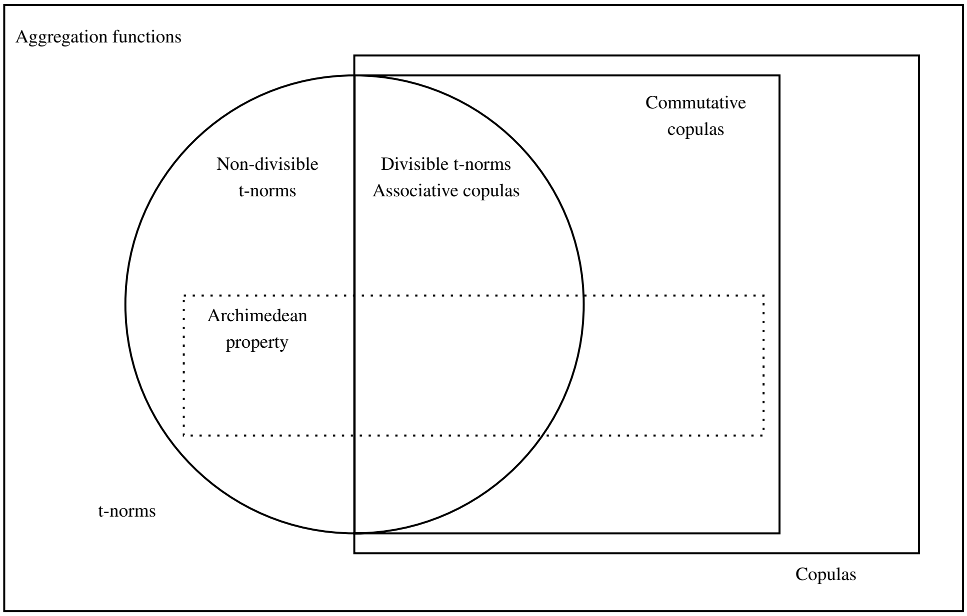

Fig. 6. Classification diagram of the different families of copulas on L_n and their relation with t-norms on L_n. Each name refers to the diagram in which it belongs, except for the Archimedean property, which cuts across all categories and is marked with dots.

5.2. Relationships between copulas and t-norms

Throughout the section it has been commented that there is a relationship between copulas and t-norms. In the next proposition the first relation is shown in terms of the divisibility.

Proposition 5.5. *([34]) All copulas on L_n are divisible and all divisible, and therefore smooth, t-norms on L_n are associative copulas on L_n.*

The main result about the relationship between copulas and t-norms is that every copula can be decomposed as a product of divisible t-norms.

Proposition 5.6. *([5,34]) Any copula on L_n is a product of associative copulas on L_n. In other words, for each copula C on L_n there exists an integer $r \geqslant 1$ and divisible t-norms $T_1, \ldots, T_r$ on L_n such that $C = T_1 \cdots T_r$.*

The associativity also relates copulas with divisible t-norms, as shown in the following proposition.

Proposition 5.7. *([34]) The following statements hold:*

- *The class of associative copulas on L_n coincides with the class of divisible (and therefore smooth) t-norms on L_n.*
- *There are 2^{n-1} associative copulas on L_n.*
- *A copula on L_n is associative if, and only if, it is an ordinal sum of Łukasiewicz copulas.*

Remark 5.2. As discussed in Remark 3.1, the associativity shown in Proposition 5.7 is related to the case in [0, 1]: a copula on [0, 1] is associative if, and only if, it is an ordinal sum of Archimedean copulas [35]. Now, keeping in mind that if a copula on L_n is associative then it is a smooth t-norm on L_n, together with Theorem 3.1, the only possibility is that it is an ordinal sum of Łukasiewicz copulas.

Finally, Fig. 6 illustrates the relationship between copulas and t-norms on L_n.

Example 5.3. The following operators are examples of copulas on L_5 belonging to the different categories in Fig. 6.

$$C_1 = \begin{array}{cc} \begin{array}{c}5\\4\\3\\2\\1\\0\end{array} & \begin{bmatrix}0&1&2&3&4&5\\0&1&1&2&3&4\\0&0&0&1&2&3\\0&0&0&0&1&2\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix} \\ & \begin{array}{cccccc}0&1&2&3&4&5\end{array}\end{array}, \quad C_2 = \begin{array}{cc} \begin{array}{c}5\\4\\3\\2\\1\\0\end{array} & \begin{bmatrix}0&1&2&3&4&5\\0&1&2&2&3&4\\0&1&1&1&2&3\\0&1&1&1&2&2\\0&1&1&1&1&1\\0&0&0&0&0&0\end{bmatrix} \\ & \begin{array}{cccccc}0&1&2&3&4&5\end{array}\end{array},$$

$$C_3 = \begin{array}{cc} \begin{array}{c}5\\4\\3\\2\\1\\0\end{array} & \begin{bmatrix}0&1&2&3&4&5\\0&0&1&2&3&4\\0&0&1&2&2&3\\0&0&1&1&1&2\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix} \\ & \begin{array}{cccccc}0&1&2&3&4&5\end{array}\end{array}.$$

- C_1 is a non-commutative copula on L_5.
- C_2 is a commutative, non-associative and non-Archimedean copula on L_5.
- C_3 is a commutative, non-associative and Archimedean copula on L_5.
- $T_{\mathbf{L}}$ is a commutative, associative (and therefore a t-norm) and Archimedean copula on L_5.
- $T_{\mathbf{M}}$ is a commutative, associative (and therefore a t-norm) and non-Archimedean copula on L_5.

5.3. Copula-related operators

In the literature we can find operators related to copulas on L_n, such as multivariate copulas [39,40], subcopulas [36] or quasi-copulas ([38]). Because quasi-copulas allow the construction from the diagonal section of another quasi-copula (see Proposition 5.11), we will focus on this family of operators.

Definition 5.3. ([38]) A *quasi-copula* Q on L_n is a binary operator $Q : L_n \times L_n \to L_n$ that satisfies the following axioms:

(Q1) $Q(x, 0) = Q(0, x) = 0$, for all $x \in L_n$.
(Q2) $Q(x, n) = Q(n, x) = x$, for all $x \in L_n$.
(Q3) Q is Lipschitz with constant 1.

According to Definition 5.3, every copula is a quasi-copula but not vice versa, so quasi-copulas are a generalization of copulas. In the following proposition a relation between both families is given:

Proposition 5.8. *([41]) A binary operator $Q : L_n \times L_n \to L_n$ is a quasi-copula on L_n if, and only if, it satisfies (Q1), (Q2) and the following property:*

(Q4) $Q(x, y) + Q(x', y') \geqslant Q(x, y') + Q(x', y)$ *for all* $x, y, x', y' \in L_n$ *such that* $x \leqslant x'$, $y \leqslant y'$ *and at least one of these four terms is equal to* 0 *or* n.

This equivalence was first proposed by Genest et al. in [42] in the [0, 1] framework, and it was Quesada-Molina et al. in [41] who translated (almost literally) the result of [42] to the L_n case. This result can be understood as a characterization of quasi-copulas; however, as the reader will see, it is simply a relation between equivalent properties. Therefore, Aguiló et al. in [43] (referring to them as irreducible quasi-copulas) proposed a characterization similar to

Proposition 5.2; now, instead of permutation matrices, another type of matrices played a fundamental role: alternating sign matrices.

Definition 5.4. [44] An $n \times n$ matrix $A = (a_{i,j})$ is an *alternating sign matrix* if it satisfies

(ASM1) $a_{i,j} \in \{-1, 0, 1\}$, for all $i, j \in \{1, \ldots, n\}$.
(ASM2) $\sum_{i=1}^{n} a_{i,k} = \sum_{j=1}^{n} a_{l,j} = 1$, for all $k, l \in \{1, \ldots, n\}$.
(ASM3) In every row and column the non-zero entries alternate in sign.

Proposition 5.9. *([43]) Let $n \geqslant 1$, and let $Q : L_n^2 \to L_n$ be a binary operator. Then, Q is a quasi-copula on L_n if, and only if, there exists an $n \times n$ alternating sign matrix $A = (a_{i,j})$ such that*

$$Q(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \sum_{i=1}^{x} \sum_{j=1}^{y} a_{i,j}, & \text{otherwise,} \end{cases} \tag{34}$$

for all $x, y \in L_n$.

Example 5.4. Let us consider the alternating sign matrix A given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the associated quasi-copula, constructed with Equation (34), is given by

$$Q = \begin{array}{c|cccccc|} 5 & 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 0 & 1 & 1 & 2 & 3 & 4 \\ 3 & 0 & 0 & 1 & 1 & 2 & 3 \\ 2 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 2 & 3 & 4 & 5 \end{array}.$$

As with other operators defined on finite chains, an important question is the cardinality of the set of quasi-copulas. Thanks to the work of Zeilberger et al. in [45] about the cardinality of alternating sign matrices, and considering the bijection of Proposition 5.9, the following result was obtained for the number of quasi-copulas.

Corollary 5.1.1. *([43,45]) Let $n \geqslant 1$, and let $\mathcal{Q}_n$ be the set of quasi-copulas defined over L_n, respectively. Then,*

$$|\mathcal{Q}_n| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{35}$$

Quasi-copulas, in addition to sharing some properties with copulas, also share the structure presented in Proposition 5.4, as shown in the next result.

Proposition 5.10. *([41]) Let Q be a quasi-copula on L_n with the set of its idempotent elements given by $I = \{0 = a_0, a_1, \ldots, a_{r-1}, a_r = n\}$ and let $J = \{1, \ldots, r\}$. Then Q is an ordinal sum $Q = \langle([a_{i-1}, a_i], Q_i)_{i \in J}\rangle$; that is, Q is given by*

$$Q(x, y) = \begin{cases} Q_i(x, y), & \text{if } (x, y) \in [a_{i-1}, a_i]^2 \text{ for some } i \in J, \\ \min\{x, y\}, & \text{otherwise,} \end{cases} \tag{36}$$

where Q_i is a quasi-copula on $[a_{i-1}, a_i]$ for each $i \in J$.

To conclude this section, we show the possibility of constructing a quasi-copula $\tilde{Q}$ from the diagonal section of another quasi-copula Q. The diagonal section of Q is the function $\delta_Q : L_n \to L_n$, defined by $\delta_Q(x) = Q(x, x)$ which satisfies:

(D1) $\delta_Q(0) = 0$ and $\delta_Q(n) = n$.
(D2) $\delta_Q(x) \leqslant \delta_Q(y)$, for all $x, y \in L_n$ such that $x \leqslant y$.
(D3) $\delta_Q(y) - \delta_Q(x) \leqslant 2(y - x)$, for all $x, y \in L_n$ such that $x \leqslant y$.
(D4) $\delta_Q(x) \leqslant x$, for all $x \in L_n$.

Let us denote by Δ_n the set of all functions $\delta : L_n \to L_n$ satisfying properties (D1)-(D4). Then we have the following result:

Proposition 5.11. *([38]) If $\delta \in \Delta_n$, then the function $\tilde{Q}$ defined by*

$$\tilde{Q}(x, y) = \min\left\{x, y, \left\lfloor \frac{\delta(x) + \delta(y)}{2} \right\rfloor\right\}, \qquad x, y \in L_n \tag{37}$$

is a quasi-copula on L_n and $\delta_{\tilde{Q}} = \delta_Q$.

The importance of this result lies in the impossibility of translating the same result for other operators, such t-norms, as it was shown in [5]; i.e., given the set $\mathcal{D}$ of functions that satisfy the diagonal properties of a t-norm, each $\delta \in \mathcal{D}$ can not be generally extended to a t-norm.

Finally, we recommend the reader to visit [38,41,46,47] for other studied properties of quasi-copulas.

6. Other aggregation functions

Some families of aggregation functions were introduced in the literature with the aim to study already known subfamilies in further detail, or to generalize other families. These families are either conjunctions, disjunctions or they can be characterized through adequate combinations of these two families of operators. The most important examples of such practice are t-subnorms, t-operators and nullnorms, which we recall briefly below.

Definition 6.1. ([48]) Let $M : L_n \times L_n \to L_n$ be a binary operator. M is a *t-subnorm on L_n* if it is associative, commutative, increasing in each argument and such that $M(x, y) \leqslant \min\{x, y\}$.

From the definition it can be deduced that t-subnorms are conjunctions and that every t-norm is a t-subnorm, but not vice versa [48]. When M is a smooth operator, we have the following relation between t-norms and t-subnorms:

Proposition 6.1. *([48]) Let M be a smooth t-subnorm on L_n. Then M is a t-norm if, and only if, $M(n, n) = n$.*

As we have seen, t-subnorms are a generalization of t-norms. Now, let us focus on a class that also generalizes t-norms with a structure similar to that of uninorms, involving adequate combinations of t-norms and t-conorms.

Definition 6.2. ([26,49]) Let $F : L_n \times L_n \to L_n$ be a binary operator. F is a *t-operator on L_n* if it is associative, commutative and increasing in each argument such that the functions $F_0(x) = F(0, x)$ and $F_n(x) = F(n, x)$ are smooth, $F(0, 0) = 0$ and $F(n, n) = n$.

In the class of t-operators, its structure has been fully characterized and it is presented in the following result:

Theorem 6.1. *([26]) A binary operator $F : L_n \times L_n \to L_n$ is a t-operator on L_n if, and only if, there exist a t-conorm S on $[0, k]$ and a t-norm T on $[k, n]$, where $k = F(0, n)$, such that F is given by*

$$F(x,y) = \begin{cases} S(x,y), & \text{if } (x,y) \in [0,k]^2, \\ T(x,y), & \text{if } (x,y) \in [k,n]^2, \\ k, & \text{otherwise.} \end{cases} \tag{38}$$

Moreover, F is smooth if, and only if, T and S are smooth operators.

Note that, on the one hand, a t-operator is a t-norm if, and only if, $k = 0$ and, on the other hand, a t-operator is a t-conorm if, and only if, $k = n$.

Finally, we introduce the family of nullnorms and a characterization of its structure which coincides with the structure of t-operators.

Definition 6.3. ([50]) Let $G : L_n \times L_n \to L_n$ be a binary operator. G is a *nullnorm on L_n* if it is commutative, associative and increasing in each argument such that there exists an absorbing element $k \in L_n$ ($G(x, k) = k$ for all $x \in L_n$) that satisfies

$$G(0,x) = x, \qquad \text{for all } x \leqslant k,$$
$$G(n,x) = x, \qquad \text{for all } x \geqslant k.$$

Theorem 6.2. *([50]) A binary operator $G : L_n \times L_n \to L_n$ is a nullnorm on L_n if, and only if, there exists $k \in L_n$, a t-conorm S on $[0, k]$ and a t-norm T on $[k, n]$ such that for all $x, y \in L_n$, G is given by*

$$G(x,y) = \begin{cases} S(x,y), & \text{if } (x,y) \in [0,k]^2, \\ T(x,y), & \text{if } (x,y) \in [k,n]^2, \\ k, & \text{otherwise.} \end{cases} \tag{39}$$

Moreover, G is smooth if, and only if, T and S are smooth operators.

From Theorems 6.1 and 6.2, as it is shown in [49], it is evident that nullnorms and t-operators are equivalent operators.

7. Implications

In this section we deal with another type of connective defined on the finite chain L_n: implication functions. Fuzzy implications are a generalization of the classical implication to fuzzy logic, finding in [51] a wide study on fuzzy implications in $[0, 1]$. For the purpose of this survey, we find in [3] and [4] some sections devoted to implications defined on L_n. Let us introduce this family of operators.

Definition 7.1. ([52]) A binary operator $I : L_n \times L_n \to L_n$ is an *implication on L_n* if, for all $x, y, z \in L_n$, it satisfies that

(I1) I is decreasing in the first argument, i.e., $I(x, y) \geqslant I(z, y)$ when $x \leqslant z$.
(I2) I is increasing in the second argument, i.e., $I(x, y) \leqslant I(x, z)$ when $y \leqslant z$.
(I3) $I(0, 0) = I(n, n) = n$ and $I(n, 0) = 0$.

Example 7.1. Some examples of implications are:

- The Łukasiewicz implication:

$$I_{\mathbf{LK}}(x,y) = \min\{n, n - x + y\}.$$

- The Godel implication:

$$I_{\mathbf{GD}}(x,y)=\begin{cases} n, & \text{if } x \leqslant y,\\ y, & \text{otherwise.}\end{cases}$$

- The Rescher implication:

$$I_{\mathbf{RS}}(x,y)=\begin{cases} n, & \text{if } x \leqslant y,\\ 0, & \text{otherwise.}\end{cases}$$

- The Fodor implication:

$$I_{\mathbf{FD}}(x,y)=\begin{cases} n, & \text{if } x \leqslant y,\\ \max\{n-x,y\}, & \text{otherwise.}\end{cases}$$

Using the property (I1), from an implication we can obtain a negation, denoted by $N_I(x)=I(x,0)$, which is called the natural negation associated to I.

Several families of implications can be obtained from different types of aggregation functions. The following are the four most important classes of implications on L_n:

- (S,N)-implications [52], defined as

$$I_{S,N}(x,y)=S(N(x),y), \qquad x,y\in L_n, \tag{40}$$

  where S and N are a t-conorm and a negation on L_n, respectively. The operator $I_{S,N}$ is always an implication, and it can be understood as the generalization of the material implication $x \Rightarrow y \equiv \neg x \vee y$. Its structure is known when S is smooth.

  Proposition 7.1. *([52]) Let S be a smooth t-conorm on L_n with the set of idempotent elements*

$$J=\{0=a_0<a_1<\cdots<a_{r-1}<a_r=n\},$$

  and let N be a negation on L_n. Then, the associated (S,N)-implication $I_{S,N}$ is given by

$$I_{S,N}(x,y)=\begin{cases} \min\{a_{i+1},N(x)+y-a_i\}, & \text{if } (N(x),y)\in[a_i,a_{i+1}]^2, \text{ with } 0\leqslant i\leqslant r,\\ \max\{N(x),y\}, & \text{otherwise.}\end{cases} \tag{41}$$

  In Equation (40), if the t-conorm is replaced with any aggregation function F such that $F(n,0)=F(0,n)=n$, the operator $I_{F,N}$ is still an implication. Nevertheless, in Table 1 this family is parametrized only with t-conorms.
- Implications generated from the idea of residuation. If T is a t-norm, the residual operator is always an implication, called the residual implication (or R-implication for short) [52], and it is defined as

$$I_T(x,y)=\max\{z\in L_n \mid T(x,z)\leqslant y\}, \qquad x,y\in L_n. \tag{42}$$

  The structure when T is smooth is given in the following proposition.

  Proposition 7.2. *([52]) Let T be a smooth t-norm on L_n with the following set of idempotent elements:*

$$J=\{0=a_0<a_1<\cdots<a_{r-1}<a_r=n\}.$$

  The associated R-implication I_T is given by

$$I_T(x,y)=\begin{cases} n, & \text{if } x\leqslant y,\\ a_{i+1}+y-x, & \text{if } a_i\leqslant y<x\leqslant a_{i+1}, \text{ with } 0\leqslant i\leqslant r,\\ y, & \text{otherwise.}\end{cases} \tag{43}$$

In Equation (42), the residuation of a t-norm can be replaced by the residuation of any aggregation function; for instance, a uninorm. However, in [53] it was shown that I_U is an implication if, and only if, U is conjunctive. See [53] for more information about its structure.

- QL-operators [54], defined by

$$I_{T,S}(x,y) = S(n-x, T(x,y)), \qquad x, y \in L_n, \tag{44}$$

where T and S are a t-norm and a t-conorm, respectively. Note that QL-operators are not implications in general (see Example 7.2), and they are called QL-implications only when they are.

Example 7.2. Let us consider the QL-operator on L_n generated by $S = S_{\mathbf{M}}$ and $T = T_{\mathbf{M}}$. In this situation, $I_{T,S}$ is not decreasing in the first argument: setting $x = n-2$, $y = n-1$ and $z = n$, we have

$$\begin{aligned} I_{T,S}(x,z) &= I_{T,S}(n-2,n) = \max\{2, \min\{n-2,n\}\} = n-2, \\ I_{T,S}(y,z) &= I_{T,S}(n-1,n) = \max\{1, \min\{n-1,n\}\} = n-1, \end{aligned}$$

for all $n \geqslant 4$.

A general characterization of those QL-operators which are also implications is still an open problem. However, the structure is known by assuming the smoothness of S and T.

Proposition 7.3. *([55]) Let T be a smooth t-norm on L_n, S a smooth t-conorm on L_n and $I_{T,S}$ its associated QL-operator on L_n. Then, the following statements are equivalent:*

1. *$I_{T,S}$ is a QL-implication on L_n.*
2. *$S(n-x, x) = n$ for all $x \in L_n$.*
3. *S is the Łukasiewicz t-conorm.*

Moreover, in this case, $I_{T,S}$ is given by $I_{T,S} = n - x + T(x,y)$, for all $x, y \in L_n$.

- D-operators [55], defined by

$$I^{T,S}(x,y) = S\left(T\left(n-x, n-y\right), y\right), \qquad x, y \in L_n, \tag{45}$$

where T and S are a t-norm and a t-conorm, respectively. In the same way as the QL-operators, not every D-operator is an implication (apply Example 7.2 using Equation (45)). Only the concrete case when T and S are smooth is characterized.

Proposition 7.4. *([55]) Let T be a smooth t-norm on L_n, S a smooth t-conorm on L_n and $I^{T,S}$ its associated D-operator on L_n. Then, $I^{T,S}$ is an implication on L_n if, and only if, S is the Łukasiewicz t-conorm. Moreover, in this case the implication is given by*

$$I^{T,S_{\mathbf{L}}}(x,y) = y + T(n-x, n-y), \tag{46}$$

for all $x, y \in L_n$.

To conclude the first part of the section, we show the close relation between QL-operators and D-operators. The result is formulated and proved for the [0, 1] case, but the proof can be adapted when the operator is defined on L_n.

Proposition 7.5. *([56]) Let T and S be a t-norm and a t-conorm on L_n, respectively. Then, the corresponding QL-operator is a QL-implication on L_n if, and only if, the corresponding D-operator on L_n is a D-implication on L_n.*

7.1. Properties of implications

There are many additional properties studied on implications depending on the context. The most usual ones are:

- Exchange principle,

$$I(x, I(y, z)) = I(y, I(x, z)), \text{ for all } x, y, z \in L_n. \tag{EP}$$

- Left neutrality principle,

$$I(n, x) = x, \text{ for all } x \in L_n. \tag{NP}$$

- Contrapositive symmetry with respect to a negation N,

$$I(x, y) = I(N(y), N(x)), \text{ for all } x, y \in L_n. \tag{CP(N)}$$

- Identity principle,

$$I(x, x) = n, \text{ for all } x \in L_n. \tag{IP}$$

- Ordering principle,

$$I(x, y) = n, \text{ if, and only if, } x \leqslant y, \text{ for all } x, y \in L_n. \tag{OP}$$

- Consequent boundary,

$$I(x, y) \geqslant y, \text{ for all } x, y \in L_n. \tag{CB}$$

- Generalized modus ponens, with respect to a t-norm T,

$$T(x, I(x, y)) \leqslant y, \text{ for all } x, y \in L_n. \tag{MP(T)}$$

- Generalized modus tollens, with respect to a t-norm T and a negation N,

$$T(N(y), I(x, y)) \leqslant N(x), \text{ for all } x, y \in L_n. \tag{MT(T,N)}$$

- Law of importation, with respect to a t-norm T,

$$I(T(x, y), z) = I(x, I(y, z)), \text{ for all } x, y, z \in L_n. \tag{LI(T)}$$

Table 1 lists the conditions that must be satisfied by the above families of implications in order to satisfy these properties.

8. Open problems and conclusions

Operators defined on finite chains represent a fundamental tool for the handling of qualitative information without the need to perform any numerical conversion between the unit interval and the finite set. In this review, the main families of aggregation functions and implications on L_n have been recalled, and also the main results on their theoretical properties have been compiled, unifying the notation existing in the literature. Despite the large number of papers devoted to the study of operators defined on finite chains, some questions remain open. The main ones are listed below:

- The cardinality of several families of operators on L_n is unknown. Although in this survey we have shown closed formulas for certain families of operators, the cardinality of the following operators remains unresolved: negations, t-norms (in general), t-conorms (whose cardinality is the same as t-norms), uninorms (in general), implications, conjunctions and disjunctions.
- Related to the previous point, the cardinality of some binary operators defined on L_n is a tough problem mainly because of the associative property. In this sense, the characterization of this property is still an open problem, although the research community has already begun to study the characterization of associativity, but only for continuous t-norms in [0, 1] (see [60,61]) and for idempotent uninorms on L_n in [13] and [30].

Table 1
Summary of conditions and properties that satisfy some families of implications on L_n. We denote with ✓ when the family always satisfies that property, with *?* when no result is known about that property in the corresponding family and with $(\star)$ when the result es equivalent to the [0, 1] case and has been translated literally.

	(S,N_1)-imp.[a]	R-imp.[b]	QL-imp.[b]	D-imp.[b]
(EP)	✓[52]	✓[52]	Iff I is (S,N_C)-imp. $(\star)$ (Prop. 8) [56]	Iff I is (S,N_C)-imp. $(\star)$ (Prop. 8) [56]
(NP)	✓[52]	✓[52]	✓ $(\star)$ (Prop. 2.6.2) [57]	✓
(CP(N))	If $N=N_1=N_C$ $(\star)$ (Prop. 2.4.3) [57]	If $N_I=N_C$ $(\star)$ (Cor. 1.5.9) [57]	Iff $I_{T',S}=I^{T',S}$ $(\star)$ (Prop. 7) [56]	Iff $I_{T',S}=I^{T',S}$ $(\star)$ (Prop. 7) [56]
(IP)	Iff $S(N(x),x)=n$ $(\star)$ (Lemma 2.4.16) [57]	✓[52]	$(\star)$ (Prop. 2.6.21 [57])	?
(OP)	If $S=S_{\mathbf{L}}$, $N_1=N_C$ [52]	✓[52]	$(\star)$ (Cor. 2.6.32 [57])	?
(CB)	✓[52]	✓[52]	?	?
(MP(T))	If $T=T_{\mathbf{L}}$ and S smooth [52,58]	If T', T are smooth, $\mathrm{Idem}_T \subseteq \mathrm{Idem}_{T'}$ [58]	If $T=T_{\mathbf{L}}$ [58]	If S smooth and $T=T_{\mathbf{L}}$ [58]
(MT(T,N))	If $T=T_{\mathbf{L}}$, S smooth and $N=N_1=N_C$ [58]	If T', T are smooth, $T=T'$ and $N=N_C$ [58]	If $T=T_{\mathbf{L}}$ and $N=N_C$ [58]	If S smooth, $T=T_{\mathbf{L}}$ and $N=N_C$ [58]
(LI(T))	Iff $T=S_N$ [59] (see Equation (14))	If T' smooth and $T=T'$ [59]	Iff $I^{T',S}$ is (S,N)-imp. and $T=S_N$ [59]	Iff $I^{T',S}$ is (S,N)-imp. and $T=S_N$ [59]
Smoothness	Iff S smooth [52]	If $T'=T_{\mathbf{L}}$ [52]	Iff T' smooth [55]	Iff T' smooth [59]

[a] We denote by N_1 the negation involved in the definition of the (S,N)-implication so that it can be distinguished from the negation N involved in (CP(N)) and (MT(T,N)).

[b] We denote by T' the t-norm involved in the definition of the R-implication, QL-implication and D-implication, so that it can be distinguished from the t-norm T involved in (MP(T)), (MT(T,N)) and (LI(T)).

- Quasi-copulas, like copulas, admit a characterization in terms of certain types of square matrices. However, it remains an open problem to try to characterize smooth conjunctions also in terms of some family of matrices, since the set of smooth conjunctions encompasses the set of quasi-copulas.
- With respect to implications, it remains an unresolved question when QL-operators and D-operators are implications in both L_n and $[0,1]$ frameworks. Furthermore, when a QL-operator or a D-operator becomes a QL-implication or a D-implication, respectively, some properties remain undetermined, as shown in Table 1. Specifically, (CB) for both families, and (IP) and (OP) for D-implications.
- Still in the field of implications, the general characterization of QL-implications, (S,N)-implications and D-implications, remains as an open problem although the structure of some of them is known under certain additional assumptions; for instance, the characterization of QL-implications with smooth underlying t-norm and t-conorm (Proposition 7.3), the characterization of (S,N)-implications when N is the only strong negation [62] or the characterization of R-implications generated from either t-norms [52] or some families of uninorms [53].

Despite all these open problems, a great advantage of operators defined on finite chains with respect to those defined on [0, 1] is that some properties are verifiable by brute force. For this reason, in order to facilitate future researches, the authors have implemented in [63] a fully-documented open-source Python library, called `DiscreteFuzzyOperators`, that integrates in the same platform different tools to work easily with operators defined on finite chains.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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